Self-scaling properties of velocity circulation in shear flows

R. Benzi, 1 L. Biferale, 2 M. V. Struglia, 3 and R. Tripiccione 4

1AIP, Via Po 14, 00100 Roma, Italy
2Department of Physics, University of Tor Vergata, Via della Ricerca Scientifica 1, I-00133 Roma, Italy
3Gruppo di Modellistica Numerica, ENEA Centro Ricerche Casaccia 110 Via Anguillarese 301, I-00060 Santa Maria di Galeria, Roma, Italy
4Istituto Nazionale di Fisica Nucleare, Sezione di Pisa, San Piero a Grado, 50100 Pisa, Italy

(Received 13 June 1996)

We investigate the scaling properties of the velocity circulation of a turbulent shear flow. We evaluate, using extended self-similarity, the circulation scaling exponents both at maximum and minimum shear regions. We show that the anomalous component of the velocity circulation and the anomalous component of the velocity structure functions are equal. [S1063-651X(97)11202-8]

PACS number(s): 47.27.-i, 05.45.+b

Fluctuations of the energy dissipation and intermittency of the velocity-field inertial-range statistics are two of the most important features of fully developed turbulent flows. A quantitative measure of intermittency is usually given by the set of scaling exponents $\xi_n$ of the $n$-order structure functions, namely,

$$F_n(r) = \langle |\delta v|^n \rangle \sim r^{\xi_n}.$$  (1)

According to the original Kolmogorov theory (K41) [1] $\xi_n = n/3$, while deviations from this law are due to intermittency corrections.

It has been argued that vortex filaments are the basic geometrical objects for describing possible dominant and subdominant contributions to the K41 power laws. In this framework, multifractal deviations to K41 have also been phenomenologically explained in terms of scaling properties of vortex filaments [2].

In this paper we investigate a possible bridge between the velocity-differences intermittency, measured by the scaling exponents of structure functions, and the scaling properties of velocity circulations around a contour $C$, namely:

$$\Gamma(C) = \oint_C \vec{\omega} \cdot d\vec{l} = \int_\Sigma \vec{\omega} \cdot d\vec{s},$$  (2)

where $\vec{\omega}$ is the vorticity field and $\Sigma$ is any surface, lying on the contour $C$. It has been emphasized that circulation is the ideal observable, able to highlight both velocity and vorticity scaling properties, and eventually linking the two statistics [3]. According to dimensional arguments, the most natural ansatz [4], is that circulation structure functions, $G_n(r)$, scale as

$$G_n(r) = \langle |\Gamma(r)|^n \rangle \sim F_n(r) r^n,$$  (3)

where $\Gamma(r)$ means the circulation evaluated around a contour of radius $r$. We investigate the validity of Eq. (3) which has been recently observed not to be satisfied [3,4].

It is of primary interest to determine whether quantities with the same physical dimension, but different tensorial structure, have the same scaling properties. It is natural to argue, for example, that in cases with strong anisotropic effects, observables with different rotational properties would have different scaling exponents [5].

In this paper we examine circulation scaling properties by using numerical data from a (3D) shear flow simulation [6]. The presence of a shear in the flow allows us to also address questions concerning the not universal character of scaling laws for anisotropic turbulence. We show that velocity difference and circulation structure functions scaling exponents have the same anomalous contribution, even if Eq. (3) is not valid.

First we briefly summarize some details of our simulation, and we present our data analysis. In order to measure the scaling properties of the circulation structure functions we shall use extended self-similarity (ESS) as recently proposed [7–9].

Our data set comes from a simulation [6] of a 3D turbulent shear flow, in a volume of $V = 160^3$ (with our choice of parameters, one lattice spacing is about one Kolmogorov scale $\eta_k$ and $R_L \sim 40$). The flow is forced such that the unstable static solution of the N-S equations is

$$U_x \sim \sin(k_z) \quad U_y = 0 \quad U_z = 0,$$  (4)

with $k_z = 8 \pi/L$ being the wave vector corresponding to the integral scales. In this way the shear has a spatial dependence $S(z) \sim \cos(k_z)$.

Some analysis of velocity statistics for the same data set have already been published [10–12]. It has been shown that the scaling exponents of the velocity structure function are strongly dependent on the presence of shear. We have evaluated $\Gamma(r)$ according to definition (2) for all squared contours with a fixed area $A = r^2$, with $r$ extending from the dissipative range to the integral scales, at two different $z$ levels corresponding to a maximum and a minimum shear level, respectively. As recently pointed out in [4], we find that the probability distribution function of $\Gamma(r)$ depends only on the area $A$ enclosed by the contour $C$, independently of the shape of the contour itself.

The scaling exponents of $G_n(r)$ are defined, in the inertial range, as...
lowing we introduce the exponents quality and the extension of the scaling regime. In the fol

Let us first mention that, similarly to what happens for the structure functions, \( F_n(r) \), due to the moderate Reynolds number of our simulation we are unable to detect a scaling law of \( G_n(r) \) with respect to the scale \( r \) [see Fig. 1, where the log-log plots of \( G_n(r) \) versus \( r \), for \( n = 2, 4, 6 \) are shown]. It is therefore useful to use ESS in order to improve the quality and the extension of the scaling regime. In the following we introduce the exponents \( \gamma_n \) defined as

\[
G_n(r) \sim [G_3(r)]^{\gamma_n}.
\]

In Fig. 2(a) we plot \( G_5 \) versus \( G_3 \) in the minimum shear zone. As one can see, a good scaling range is detected. The best fit done in the inertial range has a slope \( \gamma_5 = 1.60 \) in the dissipative range has the slope \( \gamma_5 = 5/3 \), as expected in the laminar zone from standard dimensional analysis.

Similar results have been obtained also for other structure functions. The corresponding scaling exponents are shown in Table I. These values are different from the dimensional prediction \( \gamma_n = n/3 \), giving the first positive evidence for anomalous scaling of \( G_n(r) \). In Fig. 2(b) we plot \( G_3 \) versus \( G_3 \) at the maximum shear. At variance with the analogous analysis performed on the velocity structure functions (see [6] for a detailed discussion), the circulation exhibits a wide scaling region, allowing us to give an estimate for the scaling exponent. The \( \gamma_n \) for the maximum shear case are also reported in Table I. Comparing Figs. 2(a) and 2(b), we can see that whereas at the minimum shear level the scaling region begins at few Kolmogorov scales, namely, \( 5 \eta_k \), the scaling region at the maximum shear level is smaller, beginning at about \( 9 \eta_k \) [see [6] for a discussion on this point].

Using Eq. (3), in the inertial range we obtain

\[
\chi_n = \xi_n + n.
\]

\[\gamma_1 \quad \gamma_2 \quad \gamma_4 \quad \gamma_5 \quad \gamma_6\]

\begin{tabular}{cccccc}
min sh & 0.35 & 0.68 & 1.30 & 1.60 & 1.89 \\
max sh & 0.36 & 0.69 & 1.29 & 1.56 & 1.81 \\
\end{tabular}
We have found that Eq. (8) is satisfied within 2% for all \( n, m \in [1, 6] \). This is our main result.

According to Eq. (9) and to the results so far obtained, one may argue that in the inertial range the function \( F(v/\eta_k) \) behaves as \( r^{\alpha_n} \). It follows that in the inertial range, Eq. (9) becomes

\[
H_n(r) = \mathcal{F}\left(\frac{r}{\eta_k}\right)^n.
\]

In Fig. 3, we plot \( H_6 \) versus \( H_3 \) for the minimum and maximum shear. As one can see, there is a wide scaling region extending from the smallest to the integral scale of motion. The corresponding scaling exponents have been found to satisfy the simple dimensional scaling (8) \([ d(6,3)=1.97 \) for the maximum shear and \( d(6,3)=1.98 \) for the minimum shear].

In Fig. 3, we plot \( H_6 \) versus \( H_3 \) for the minimum and maximum shear. The best fits in the figure correspond to pending on the shear. The best fits in the dissipative range and to the best fit in the inertial range. In Fig. 4(a) we show, in a log-log plot, the ratio \( H_n(r) \) versus \( r \) for \( n=5 \). We can easily recognize two scaling regions: the first one is in the dissipative region, where the best fit has a slope close to 5, the second one is in the inertial range, with a slope 3.20. In Fig. 4(b) we plot the exponents \( \alpha(n) \) vs \( n \) for the minimum and maximum shear. In both cases \( \alpha(n) \) falls on a straight line \( \alpha(n)=\alpha_n, n \), with \( \alpha_n \) depending on the shear. The best fits in the figure correspond to \( \alpha_n=0.68 \) for the minimum shear and \( \alpha_n=0.54 \) for the maximum shear. We have no clear explanation for the dependency of \( \alpha_n \) on the shear strength.

In this paper we have mainly investigated the self-scaling properties of the velocity circulation. Indeed at the maximum shear, the velocity-field structure functions has a poor scaling behavior [6], whereas the scaling of \( G_n(r) \) vs \( G_3 \) is much clearer. Within numerical error we do not see any strong differences between scaling properties of circulation structure functions at minimum and maximum shear.

From Fig. 4(b) we conclude that the anomalous scaling of velocity circulation is equal to the anomalous scaling of velocity structure functions, in the sense that

\[
\chi_n - \xi_n = \alpha_n, n.
\]

Figure 4(b) shows that \( \alpha_n \) is a nonuniversal quantity, its value may depend on geometrical constraints. One may argue, that such dependency is due to the stretching and folding of vorticity structures induced by the shear. Nevertheless, because the nonlinear dependency of \( \chi_n \) from \( n \) is always the same, one may argue that the analysis of intermittency, in terms of the scaling exponents of the velocity circulation, does not need different physical interpretations with respect to those already proposed for the velocity structure function.