Kinetic and Related Models
Volume **X**, Number **0X**, XX **200X**

American Institute of Mathematical Sciences
pp. **X–XX**

VALIDITY OF THE BOLTZMANN EQUATION WITH AN EXTERNAL FORCE

Raffaele Esposito

Dipartimento di Matematica, Università di L'Aquila,
Coppito, 67100 AQ, Italy

Yan Guo

Division of Applied Mathematics, Brown University,
Providence, RI 02812, U.S.A.

Rossana Marra

Dipartimento di Fisica and Unità INFN, Università di Roma Tor Vergata, 00133 Roma, Italy
00133 Roma, Italy

(Communicated by the associate editor name)

Abstract. We establish local-in-time validity of the Boltzmann equation in the presence of an external force deriving from a C^2 potential.

1. **Introduction.** In many applications of the Boltzmann equation the presence of an external force plays a crucial role. As an example, among many others, we mention the treatment of the Benard problem [9, 1] where the presence of the gravity is essential. A lot of activity has been devoted recently to establish the wellposedness of the initial boundary value problem in the presence of a given force [14, 6, 7]. Self consistent forces arise in the Boltzmann equation, when long range interactions are included in the model and treated as mean field terms. This is the case, for example, of the Vlasov-Boltzmann [10, 11] and Vlasov-Maxwell-Boltzmann equations [12], as well as kinetic systems undergoing phase transitions [2, 4, 8].

We start with a system of N spheres of diameter ε undergoing elastic collisions when in contact. One can derive the corresponding BBGKY hierarchy of equations for j-particle distribution functions $P^j_{N,\varepsilon}$ for $1 \le j \le N$. The Boltzmann-Grad limit is the scaling limit such that N goes to infinity and ε goes to 0, but the product

$$N\varepsilon^{d-1} \to \lambda \ \ \text{(finite constant)}$$

(λ is the inverse of the mean free path). The goal is to show that, under an initial independence assumption, for any j, $P^j_{N,\varepsilon} \to f_j$ in a suitable sense, in the Boltzmann-Grad limit, where f_j are solutions to the Boltzmann hierarchy, which are naturally factorized as j-product of solutions to the Boltzmann equation (propagation of chaos).

Lanford's strategy proof of the validity of the Boltzmann equation [13] is based essentially on four steps: 1) Series representation of solutions to both BBGKY hierarchy and Boltzmann hierarchy (for details on the two hierachies we refer to [5]

2000 *Mathematics Subject Classification.* Primary: 76P05, 82C40; Secondary: 76A20.
Key words and phrases. Rarified Gases, Boltzmann Equation, Derivation, External Force.

and references quoted therein) in terms of a sum over graphs representing backward in time collision histories (Sections 2 and 3); 2) Observe that factorized initial data for the Boltzmann hierarchy produce factorized solutions each factor solving the Boltzmann equation; 3) Absolute convergence with respect to summation over N for each series graph representation to BBGKY and Boltzmann hierarchies for short times (Section 4); 4) Each term in BBGKY graph series converges to the corresponding term in the Boltzmann hierarchy as $\varepsilon \to 0$ (Section 5).

In Lanford's original proof, there is no external field acting on these particles. Of all these steps, the only one which is seriously affected by the presence of the force is the fourth one. It has been observed in [3] that for the Lorentz gas in the presence of a magnetic field, the corresponding Boltzmann equation has not to be expected to correctly describe the behavior of the system in the Boltzmann-Grad limit. The problem is the following: in the series representing the solution to the BBGKY hierarchy, each graph contains originally j particles at time t to which are added n new particles at previous times $t_1 > t_2 \cdots > t_n$, representing the n collisions described by the graph (see Section 3 for details). Roughly speaking, the j particle existing at time t evolve backward in time, according to the j-particle dynamics up to time t_1 when one of the j particles collides with a new particle. Then the $j+1$ particles present at time t_1 evolve up to the time t_2 according to the $j+1$-particle dynamics and so on. The difference when one considers the Boltzmann hierarchy is that in this case, after each collision the particles evolve independently according to the one-particle dynamics. If two particles collide again, after the first collision, in the true dynamics, this event, called *recollision* prevents the convergence to the corresponding term for the Boltzmann hierarchy because the recollision does not occur in the free evolution. Fortunately in absence of forces, it can be shown that the set of initial data which evolve into recollisions has a Lebesgue measure which goes to 0 as $\varepsilon \to 0$ for any $t > 0$ and hence it is possible to prove the almost everywhere convergence of the solution of the BBGKY hierarchy to the solution of the Boltzmann hierachy. When there is an external force, this is false without extra assumptions, because a new type of recollisions appears. As matter of fact, as discussed in Remark 5.2 below, if the force is harmonic, because of the periodicity of the motion, recollision will occur with positive probability after a period. Therefore, in this case the validity cannot be proved, along the Lanford strategy, for times longer than the period.

On the other hand, it is clear that, for short times, the effect of the force, which is quadratic in time, should not change too much the straight line motion of each particle, which, after a collision tends to separate linearly in time. Our main contribution is to demonstrate there exists fixed time interval, independent of N and ε, such that the curved trajactories cannot re-collide within such a period of time, at least for a set of full Lebesgue measure as $\varepsilon \to 0$. This is true if the force if uniformly Lipshitz continuous and it is the content of Theorem 5.1 below. Based on this we can prove the validity of the Bolztmann equation in the presence of an uniformly Lipshitz continuous force, in the same sense of Lanford. The precise statement is given in Theorem 5.2 below.

2. **BBGKY Hierarchy.** We consider a system of N hard spheres of diameter ε moving in $\mathbb{R}^d$ under the action of a external smooth force F and undergoing elastic collisions when two sphere are in contact. To be more precise, let $\Lambda \subset \mathbb{R}^d$ be an open cubic box of size L (the parameter Λ will be kept fixed in this note and can

be conveniently assumed to be 1). The N-particle phase space Γ_ε^N is defined as follows: with the notation $Z_N = (X_N, V_N) = (x_1, v_1, \dots, x_N, v_N)$, where x_i is the center of the sphere i and v_i its velocity,

$$\Gamma_\varepsilon^N = \{Z_N \in (\Lambda \times \mathbb{R}^d)^N \,|\, |x_\ell - x_k| > \varepsilon \quad \forall \ell \neq k\}.$$

The evolution of the system is defined as follows: given $Z_N \in \Gamma_\varepsilon^N$, we define

$$T_t^{N,\varepsilon} Z_N = (x_1(t), v_1(t), \dots, x_N(t), v_N(t)), \tag{2.1}$$

with

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = -\nabla U(x_i(t)), \end{cases} \qquad i = 1, \dots, N, \tag{2.2}$$

with initial data

$$x_i(0) = x_i, \quad v_i(0) = v_i \tag{2.3}$$

for any time t such that $T_t^{N,\varepsilon} Z_N \in \Gamma_\varepsilon^N$. We assume that the force F is uniformly Lipshitz continuous, so that the Cauchy problem for the system (2.2) is well-posed, as long as the spheres do not touch each other or the boundary

In order to complete the definition of the evolution $T_t^{N,\varepsilon}$, we prescribe elastic collisions with the boundary or when two particles are at distance ε. If the sphere i is at the position $x_i(t) \in \partial\Lambda$, with velocity $v_i(t)$ at a certain time t, its velocity has a jump and its value at time $t^+ = t + 0^+$ is given by

$$v_i(t^+) = v_i(t) - 2(n(x_i(t)) \cdot v_i) n(x_i(t)) \tag{2.4}$$

where $n(x)$ is the inner normal to $\partial\Lambda$ in x. When a couple of particles ℓ and k are such that at a certain time t $|x_\ell(t) - x_k(t)| = \varepsilon$, the velocities at time t^+ are defined as

$$\begin{aligned} v_\ell(t^+) &= v_\ell(t) - [n_{\ell,k} \cdot (v_\ell - v_k)] n_{\ell,k} \\ v_k(t^+) &= v_k(t) + [n_{\ell,k} \cdot (v_\ell - v_k)] n_{\ell,k} \end{aligned}$$

where

$$n_{\ell,k} = \frac{x_k(t) - x_\ell(t)}{|x_k(t) - x_\ell(t)|}, \tag{2.5}$$

provided that the two particles are approaching to each other, meaning that

$$n_{\ell,k} \cdot (v_\ell(t) - v_k(t)) \leq 0. \tag{2.6}$$

Otherwise, nothing happens. The evolution after t^+ continues according to (2.2), with the new initial data, till the next time when two particles are in contact or a particle hits the boundary. At that time the same prescription is applied. Therefore, the above evolution is well defined at any time because the force is uniformly Lipshitz continuous. Clearly, with the same procedure the evolution $T_t^{N,\varepsilon}$ can be defined also for $t < 0$.

We do not define the dynamics when more than two particles are in contact. This and other pathological situations are easily shown to have zero Lebesgue measure and will be therefore irrelevant in the following. We refer to [5] for details.

The collisions both between particles and at the boundary, are defined in such a way to preserve the kinetic energy:

$$\frac{1}{2}|v_i(t^+)|^2 = \frac{1}{2}|v_i(t)|^2 \tag{2.7}$$

at the boundary, and

$$\frac{1}{2}|v_\ell(t^+)|^2 + \frac{1}{2}|v_k(t)|^2 = \frac{1}{2}|v_\ell(t)|^2 + \frac{1}{2}|v_k(t^+)|^2 \tag{2.8}$$

in binary collisions. The potential U is assumed smooth, with bounded second derivatives. We also assume that there is $B > 0$ such that

$$U(x) \geq -B \quad \text{for all } x \in \Lambda. \tag{2.9}$$

The total energy is defined as

$$\frac{1}{2}|v|^2 + U(x). \tag{2.10}$$

The energy of the particle i, $\frac{1}{2}|v(t)|^2 + U(x(t))$. is conserved, even during collisions with the boundary, as long as it does not undergoes collisions with other particles. During a binary collision the sum of the kinetic energies of the two particles involved is conserved, while the potential energy of each particle does not change. As a consequence, the total energy $E(Z_N) = \sum_{i=1}^N \frac{1}{2}|v_i|^2 + U(x_i)$is conserved during the evolution $T_t^{N,\varepsilon}$

$$E(T_t^{N,\varepsilon} Z_N) = E(Z_N). \tag{2.11}$$

Let $P_{N,\varepsilon}$ be a probability measure on Γ_ε^N invariant under permutations and absolutely continuous with respect to the Lebesgue measure with a density still denoted by $P_{N,\varepsilon}(Z_N)$. The probability measure

$$P_{N,\varepsilon}(t) = P_{N,\varepsilon} \circ T_{-t}^{N,\varepsilon} \tag{2.12}$$

is also absolutely continuous with respect to the Lebesgue measure with a density $P_{N,\varepsilon}(Z_N, t)$ satisfying the Liouville equation

$$\partial_t P_{N,\varepsilon}(t) + \sum_{i=1}^N \Big[v_i \cdot \nabla_{x_i} P_{N,\varepsilon}(t) - \nabla_{x_i} U(x_i) \cdot \nabla_{v_i} P_{N,\varepsilon}(t) \Big] = 0 \tag{2.13}$$

in Γ_e^N, with initial condition

$$P_{N,\varepsilon}(Z_N, 0) = P_{N,\varepsilon}(Z_N) \tag{2.14}$$

and boundary conditions

$$P_{N,\varepsilon}(x_1, v_1, \dots, x_i, v_i, \dots, x_N, v_N, t) = P_{N,\varepsilon}(x_1, v_1, \dots, x_i, \tilde{v}_i, \dots, x_N, v_N, t),$$
$$\text{if } x_i \in \partial\Lambda, \text{ with } \tilde{v}_i = v_i - 2[n(x_i) \cdot v_i] n(x_i); \tag{2.15}$$

$$P_{N,\varepsilon}(x_1, v_1, \dots, x_i, v_i, \dots, x_j, v_j, \dots, x_N, v_N, t) =$$
$$P_{N,\varepsilon}(x_1, v_1, \dots, x_i, v_i', \dots, x_j, v_j', \dots, x_N, v_N, t), \qquad \text{if } |x_i - x_j| = \varepsilon, \tag{2.16}$$

with

$$v_i' = v_i - n_{i,j}[\cdot(v_i - v_j)] n_{i,j}, \quad v_j' = v_j - n_{i,j}[\cdot(v_i - v_j)] n_{i,j}, \quad n_{i,j} = \frac{x_j - x_i}{|x_j - x_i|}.$$

The equation (2.12) defines a map on the probability distributions that we denote it by $S^{k,\varepsilon}(t)$: For any probability density P on Γ_ε^k and any $Z_k \in \Gamma_\varepsilon^k$, we set

$$S^{k,\varepsilon}(t) P(Z_k) = P(T_{-t}^{k,\varepsilon} Z_k). \tag{2.17}$$

The j-particle probability density $P_{N,\varepsilon}^{(j)}$ is obtained from $P_{N,\varepsilon}$ by integrating $P_{N,\varepsilon}$ over the positions and velocities of $N-j$ particles. By the permutation invariance we can choose to integrate over the last $N-j$ positions and velocities, so

$$P_{N,\varepsilon}^{(j)}(Z_j) = \int_{\Gamma_\varepsilon^{N-j}(Z_j)} dZ^{N-j} P_{N,\varepsilon}(Z_j \cup Z^{N-j}), \tag{2.18}$$

where $Z^{N-j} = (x_{j+1}, v_{j+1}, \dots, x_N, v_N)$ and $Z_j \cup Z^{N-j} = (x_1, v_1, \dots, x_j, v_j, x_{j+1}, v_{j+1}, \dots, x_N, v_N)$. The domain of integration in the above definition is

$$\Gamma_\varepsilon^{N-j}(Z_j) = \{W \in (\Lambda \times \mathbb{R}^d)^{N-j} \,|\, Z_j \cup W \in \Gamma_\varepsilon^N\}. \tag{2.19}$$

The j-particle probability densities satisfy the BBGKY hierachy

$$\partial_t P_{N,\varepsilon}^{(j)}(Z_j, t) + \sum_{i=1}^{j} \Big[v_i \cdot \nabla_{x_i} P_{N,\varepsilon}^{(j)}(t) - \nabla_{x_i} U(x_i) \cdot \nabla_{v_i} P_{N,\varepsilon}^{(j)}(t) \Big] = C_{N,\varepsilon}^{(j+1)} P_{N,\varepsilon}^{(j+1)}(Z_j, t), \tag{2.20}$$

in Γ_ε^j, with boundary conditions similar to those for $P_{N,\varepsilon}$. The collision operator $C_{N,\varepsilon}^{(j+1)}$ maps $(j+1)$-particle probability densities in into j-particle probability densities as follows:

$$C_{N,\varepsilon}^{(j+1)} P_{N,\varepsilon}^{(j+1)} = \sum_{i=1}^{j} C_{N,\varepsilon}^{(j+1),i,+} P_{N,\varepsilon}^{(j+1)} - \sum_{i=1}^{j} C_{N,\varepsilon}^{(j+1),i,-} P_{N,\varepsilon}^{(j+1)} \tag{2.21}$$

with

$$\begin{aligned} C_{N,\varepsilon}^{(j+1),i,+} P_{N,\varepsilon}^{(j+1)}(Z_j) &= (N-j)\varepsilon^{d-1} \int_{\mathbb{R}^d} dv_{j+1} \int_{S_+^{d-1}(v_i - v_{j+1})} d\omega \\ &|\omega \cdot (v_i - v_{j+1})| P_{N,\varepsilon}^{(j+1)}(x_1, v_1, \dots, x_i, v_i', \dots, x_j, v_j, x_i - \varepsilon\omega, v_{j+1}'), \end{aligned} \tag{2.22}$$

$$\begin{aligned} C_{N,\varepsilon}^{(j+1),i,-} P_{N,\varepsilon}^{(j+1)}(Z_j) &= (N-j)\varepsilon^{d-1} \int_{\mathbb{R}^d} dv_{j+1} \int_{S_+^{d-1}(v_i - v_{j+1})} d\omega \\ &|\omega \cdot (v_i - v_{j+1})| P_{N,\varepsilon}^{(j+1)}(x_1, v_1, \dots, x_i, v_i, \dots, x_j, v_j, x_i + \varepsilon\omega, v_{j+1}), \end{aligned} \tag{2.23}$$

where

$$S_+^{d-1}(V) = \{\omega \in \mathbb{R}^d \,|\, |\omega| = 1,\, V \cdot \omega \geq 0\} \tag{2.24}$$

and v_i' and v_{j+1}' are related to v_i and v_{j+1} by the relations

$$v_i' = v_i - \omega[\cdot(v_i - v_{j+1})]\omega, \quad v_{j+1}' = v_{j+1} - \omega[\cdot(v_i - v_{j+1})]\omega. \tag{2.25}$$

It is convenient to write (2.20) in a time integrated (mild) form:

$$P_{N,\varepsilon}^{(j)}(t) = S^{j,\varepsilon}(t) P_{N,\varepsilon}^{(j)}(0) + \int_0^t dt_1 S^{j,\varepsilon}(t - t_1) C_{N,\varepsilon}^{(j+1)} P_{N,\varepsilon}^{(j+1)}(t_1). \tag{2.26}$$

By iterating (2.26) $N-j$ times, we obtain the solution to the BBGKY hierarchy in the form of the *finite* sum:

$$P_{N,\varepsilon}^{(j)}(t) = \sum_{n \geq 0} \mathcal{I}_{N,\varepsilon}^{j,n}(t), \tag{2.27}$$

$$\mathcal{I}_{N,\varepsilon}^{j,n}(t) = \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \mathcal{K}_{N,\varepsilon}^{j,n}(t, t_1, \dots, t_n) \tag{2.28}$$

$$\begin{aligned}\mathcal{K}^{j,n}_{N,\varepsilon}(t,t_1,\dots,t_n) = S^{j,\varepsilon}(t-t_1)C^{(j+1)}_{N,\varepsilon}S^{j+1,\varepsilon}(t_1-t_2)C^{(j+2)}_{N,\varepsilon}\\ \dots S^{j+n-1,\varepsilon}(t_{n-1}-t_n)C^{(j+n)}_{N,\varepsilon}S^{j+n,\varepsilon}(t_n)P^{(j+n)}_{N,\varepsilon}(0),\end{aligned} \tag{2.29}$$

with the convention that $t_0 = t$ and for $n = 0$ there is no integration; moreover, we $P^{(j)}_{N,\varepsilon} = 0$ for $j > N$.

3. **Boltzmann Hierarchy.** Our aim is to take the limit of the above expansion as $N \to \infty$ and $\varepsilon \to 0$. From the expressions (2.22) and (2.23) it is clear that we need to assume $N\varepsilon^{d-1} = \lambda^{-1}$ finite. The number $\lambda > 0$ is related to the mean free path. This limit is the Boltzmann-Grad limit (see for example [5]). The limiting equation for the 1-particle density distribution $f_1(x, v, t) = f(x, v, t)$ in this limit is expected to be the Boltzmann equation

$$\partial_t f + v\cdot\nabla_x f + F(x)\cdot\nabla_v f = Q(f) \tag{3.1}$$

where $Q(f)$ is the collision operator

$$Q(f)(v) = \int_{\mathbb{R}} dv_* \int_{S^{d-1}(v-v_*)} d\omega|\omega\cdot(v-v_*)|\{f'f'_* - ff_*\}, \tag{3.2}$$

with $f' = f(v')$, $f'_* = f(v'_*)$, $f_* = f(v_*)$ and $f = f(v)$, with v' and v'_* the incoming velocities of a collision process with outgoing velocies v and v_* and impact parameter ω:

$$v' = v - \omega(\omega\cdot(v-v_*)), \quad v'_* = v_* + \omega(\omega\cdot(v-v_*)). \tag{3.3}$$

Below we will be mainly interested into the so called *mild* form of the Boltzmann equation, i.e. its time integrated version: Introduce the notation $T_t z$ for the free flow in $\Lambda\times\mathbb{R}^d$ under the action of the force F and with elastic collisions at $\partial\Lambda$, namely $T_t z = (x(t), v(t))$ with

$$\begin{aligned}\dot x &= v\\ \dot v &= F(x)\end{aligned}$$

as long as $x(t)\in\Lambda$ and $v(t^+) = v(t) - 2n(x(t)(v(t)\cdot n(x(t))$ if $x(t)\in\partial\Lambda$. Moreover set $S(t)f(z) = f(T_{-t}z)$. Then the *mild form* is given by

$$f(t) = S(t)f(0) + \int_0^t d\tau S(t-\tau)Q(S(\tau)f) \tag{3.4}$$

In order to make the comparison between the sum (2.29) and the solution to the Boltzmann equation, we introduce the *Boltzmann Hierarchy*:

Let $\mathbf{f}(t) = \{f_j(t)\}_{j\in\mathbb{N}}$ be a family of positive probabilities densities, such that for each $j\ge 1$, $f_j(t)$ is a probability density on $(\Lambda\times\mathbb{R}^d)^j$ and

$$\int_{(\Lambda\times\mathbb{R}^d)^j} f_j(x_1,v_1,\dots,x_j,v_j,t)dx_j dv_j = f_{j-1}(x_1,v_1,\dots,x_{j-1},v_{j-1},t) \tag{3.5}$$

with the convention $f_0 = 1$. The Boltzmann Hierarchy reads:

$$\begin{aligned}\partial_t f_j(Z_j,t) + \sum_{i=1}^j \Big[v_i\cdot\nabla_{x_i} f_j(Z_j,t) - \nabla_{x_i}V(x_i)\cdot\nabla_{v_i} f_j(Z_j,t)\Big]\\ = C^{(j+1)} f_{j+1}(Z_j,t),\end{aligned} \tag{3.6}$$

for any $t > 0$ and $Z_j \in (\Lambda \times \mathbb{R}^d)^j$. The collision operator $C^{(j+1)}$ is defined as:

$$C^{(j+1)} f_{j+1} = \sum_{i=1}^{j} C^{(j+1),i,+} f_{j+1} - \sum_{i=1}^{j} C^{(j+1),i,-} f_{j+1} \tag{3.7}$$

with

$$C^{(j+1),i,+} f_{j+1}(Z_j) = \lambda^{-1} \int_{\mathbb{R}^d} dv_{j+1} \int_{S_+^{d-1}(v_i - v_{j+1})} d\omega \\ |\omega \cdot (v_i - v_{j+1})| f_{j+1}(x_1, v_1, \dots, x_i, v_i', \dots, x_j, v_j, x_i, v_{j+1}'), \tag{3.8}$$

$$C^{(j+1),i,-} f_{j+1}(Z_j) = \lambda^{-1} \int_{\mathbb{R}^d} dv_{j+1} \int_{S_+^{d-1}(v_i - v_{j+1})} d\omega \\ |\omega \cdot (v_i - v_{j+1})| f_{j+1}(x_1, v_1, \dots, x_i, v_i, \dots, x_j, v_j, x_i, v_{j+1}), \tag{3.9}$$

The connection between the Boltzmann equation and the Boltzmann Hierarchy is based on the following remark: If $f^0(x, v)$ is any probability density on $\Lambda \times \mathbb{R}^d$, $f(x, v, t)$ is a solution to the Boltzmann equation with initial datum f^0 and we define, for any $j > 0$,

$$f_j^0(x_1, v_1, \dots, x_j, v_j) = \prod_{i=1}^{j} f^0(x_i, v_i), \tag{3.10}$$

then the sequence $\mathbf{f}(t) = \{f_j(x_1, v_1, \dots, x_j, v_j, t)\}_{j \in \mathbb{N}}$, with

$$f_j(x_1, v_1, \dots, x_j, v_j, t) = \prod_{i=1}^{j} f(x_i, v_i, t), \tag{3.11}$$

is solution to the Boltzmann Hierachy.

Also the solution to the Boltzmann Hierarchy is conveniently written in integral form: We set $T_t^j Z_j = (T_t z_1, \dots, T_t z_j)$ and

$$S^j(t) f_j(Z_j) = f_j(T_{-t}^j Z_j). \tag{3.12}$$

The Boltzmann Hierarchy is equivalent to

$$f_j(t) = S^j(t) f_j(0) + \int_0^t dt_1 S^j(t - t_1) C^{(j+1)} f_{j+1}(t_1). \tag{3.13}$$

By iterating (3.13) $N - j$ times, we obtain the solution to the Boltzmann hierarchy in the form of the *infinite* sum:

$$f_j(t) = \sum_{n \geq 0} \mathcal{I}^{j,n}(t), \tag{3.14}$$

$$\mathcal{I}^{j,n}(t) = \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \mathcal{K}^{j,n}(t, t_1, \dots, t_n), \tag{3.15}$$

$$\mathcal{K}^{j,n}(t, t_1, \dots, t_n) = S^j(t - t_1) C^{(j+1)} S^{j+1}(t_1 - t_2) C^{(j+2)} \\ \dots S^{j+n-1}(t_{n-1} - t_n) C^{(j+n)} S^{j+n}(t_n) f_{j+n}(0). \tag{3.16}$$

The Lanford proof [13] is based on bounding for short times all the terms of the sums (2.27) and (3.14) by an absolutely convergent series, uniformly in N, and proving for each fixed n, the convergence of each term of the sum in (2.27) to a corresponding term in the series (3.14). Since the book-keeping of all the

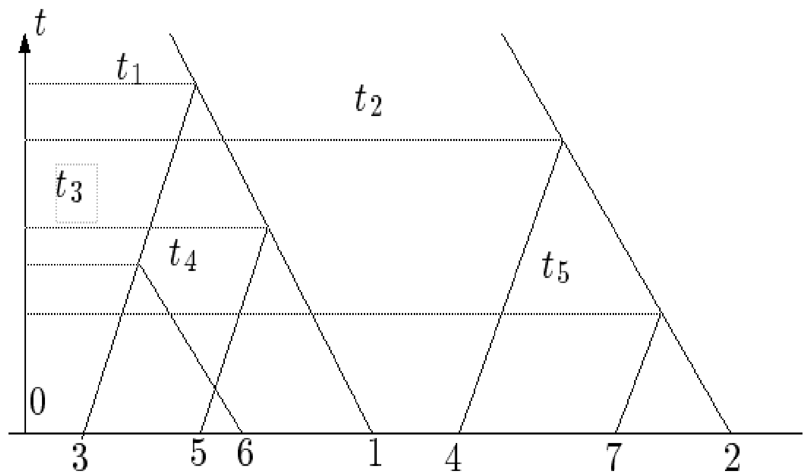

FIGURE 1. Tree $G(2,5) = 1,2,1,3,2$

contributions in the series is rather complex, it is convenient to introduce a graphical representation by associating to each of the terms of the sums suitably defined graphs.

To this end, we first exploit the expression of the collision operators in (2.29) (the analysis for (3.16) is similar). We have:

$$P^{(j)}_{N,\varepsilon}(t) = \sum_{n\ge 0}\sum_{\underline{\sigma}_n}\sum_{k_1,\dots,k_n}{}'(-1)^{|\underline{\sigma}_n|}\int_0^t dt_1\int_0^{t_1}dt_2\dots\int_0^{t_{n-1}}dt_n$$
$$S^{j,\varepsilon}(t-t_1)C^{(j+1),k_1,\sigma_1}_{N,\varepsilon}S^{j+1,\varepsilon}(t_1-t_2)C^{(j+2),k_2,\sigma_2}_{N,\varepsilon}\dots S^{j+n-1,\varepsilon}(t_{n-1}-t_n)$$
$$C^{(j+n),k_n,\sigma_n}_{N,\varepsilon}S^{j+n,\varepsilon}(t_n)P^{(j+n)}_{N,\varepsilon}(0), \tag{3.17}$$

where $\underline{\sigma}_n = (\sigma_1,\dots,\sigma_n)$ with $\sigma_k = \pm 1$ and $|\underline{\sigma}_n| = \sum_{k=1}^n \sigma_n$. Moreover,

$$\sum_{k_1,\dots,k_n}{}' = \sum_{k_1=1}^{j}\sum_{k_2=1}^{j+1}\cdots\sum_{k_n=1}^{j+n}. \tag{3.18}$$

Fixed j and n, to each choice of $k_1 \in \{1,\dots,j\}$, $k_2 \in \{1,\dots j+1\}$, $\dots$, $k_n \in \{1,\dots,j+n\}$ we associate a graph $G(j,n)$, which we represent as a tree. The tree has j roots representing the j particles existing at time t, each of the n branches represents the creation of a new particle at the time t_ℓ chosen among $t_1 > t_2 > \dots > t_n$ near the particle k_ℓ, with incoming or outgoing velocities, depending on σ_ℓ. The straight lines represent the backward evolution of the particles between creation of new particles. Note that with an external force, the real trajectories are not straight lines, so the graph has to be intended just symbolically. An example of tree with $j = 2$ and $n = 5$ is given in the Figure 1. The set of all such graphs is denoted by $\mathcal{G}(j,n)$ and the sum $\sum_{k_1,\dots,k_n}{}'$ becomes $\sum_{G(j,n)\in\mathcal{G}(j,n)}$.

We denote by $\tilde{Z}^\varepsilon(s)$ the positions and velocities at the time s in a given tree. If $s \in (t_{\ell+1,t_\ell})$, then it contains the positions and velocities of just $j+\ell$ particles, denoted by $(\tilde{x}_1(s),\tilde{v}_1(s),\dots,\tilde{x}_{j+\ell}(s),\tilde{v}_{j+\ell}(s))$. The particle $j+\ell$ is created at time t_ℓ by the particle k_ℓ in the position $\tilde{x}_{j+\ell}(t_\ell) = \tilde{x}_{k_\ell}(t_\ell) - \sigma_\ell\omega_\ell\varepsilon$, with velocity $\tilde{v}(t_{j+\ell}) =$

$v_{j+\ell}$ if $\sigma_\ell = -1$ and with velocity $\tilde{v}(t^-_{j+\ell}) = v_{j+\ell} - \omega_\ell(\omega_\ell \cdot (\tilde{v}(t_{k_\ell}) - v_{j+\ell}))$, if $\sigma_\ell = 1$. Note that, since we are looking at the backward evolution, the velocity has to be defined at the time $t^-_\ell = t_\ell + 0^-$. In the interval $(t_\ell, t_{\ell+1})$ the particle evolve according to $T^{j+\ell,\varepsilon}_{-t}$, thus the internal collisions between the $j+\ell$ particles are still to be taken into account. To be consistent with this notation, we set $t_0 = t$, $t_{n+1} = 0$ and, when $\ell = 0$, so that there are only the j initial particles, we set $(\tilde{x}_1(s), \tilde{v}_1(s), \dots, \tilde{x}_j(s), \tilde{v}_j(s)) = (x_1(s), v_1(s), \dots, x_j(s), v_j(s))$.

We also use the notation

$$\mathbf{t}_n = (t_1, \dots, t_n),$$
$$\underline{\omega}_n = (\omega_1, \dots, \omega_n),$$
$$\mathbf{v}_{j,n} = (v_{j+1} \dots, v_{j+n}),$$

and introduce the measure

$$d\mu(\mathbf{t}_n, \underline{\omega}_n, \mathbf{v}_n) = \chi(\{t > t_1 > t_2 \dots > t_n\}) dt_1 \dots dt_n d\omega_1 \dots d\omega_n dv_{j+1} \dots dv_{j+n}, \tag{3.19}$$

so that (3.17) can be written as

$$P^{(j)}_{N,\varepsilon}(t)(Z_j) = \sum_{n \geq 0} \alpha_n(j) \sum_{\underline{\sigma}_n} \sum_{G(j,n) \in \mathcal{G}(j,n)} (-1)^{|\underline{\sigma}_n|} \int d\mu(\mathbf{t}_n, \underline{\omega}_n, \mathbf{v}_n) B(\mathbf{t}_n, \underline{\omega}_n, \mathbf{v}_{j,n}) P^{(j+n)}_{N,\varepsilon}(\tilde{Z}^\varepsilon(0), 0), \tag{3.20}$$

where

$$\alpha_n(j) = (N-j)(N-j-1) \times (N-j-n+1)\varepsilon^{(d-1)n} \tag{3.21}$$

and

$$B(\mathbf{t}_n, \underline{\omega}_n, \mathbf{v}_{j,n}) = \prod_{\ell=1}^n |\omega_\ell \cdot (\tilde{v}_{k_\ell}(t_\ell) - v_{j+\ell})|. \tag{3.22}$$

A similar procedure can be used to represent the terms in (3.16). The result is

$$f_j(t)(Z_j) = \sum_{n \geq 0} \lambda^{-n} \sum_{\underline{\sigma}_n} \sum_{G(j,n) \in \mathcal{G}(j,n)} (-1)^{|\underline{\sigma}_n|} \int d\mu(\mathbf{t}_n, \underline{\omega}_n, \mathbf{v}_n) B(\mathbf{t}_n, \underline{\omega}_n, \mathbf{v}_{j,n}) f_{j+n}(\bar{Z}(0), 0), \tag{3.23}$$

the main change being the definition of $\bar{Z}(s)$. Indeed given a trees as before, $\bar{Z}(s)$ are the positions and velocities at the time s in the given tree. If $s \in (t_{\ell+1}, t_\ell)$, then it contains the positions and velocities of just $j+\ell$ particles, denoted by $(\bar{x}_1(s), \bar{v}_1(s), \dots, \bar{x}_{j+\ell}(s), \bar{v}_{j+\ell}(s))$. The particle $j+\ell$ is created at time t_ℓ by the particle k_ℓ in the position $\bar{x}_{j+\ell}(t_\ell) = \tilde{x}_{k_\ell}(t_\ell)$, thus exactly where the ancestor particle is, with velocity $\bar{v}(t_{j+\ell}) = v_{j+\ell}$ if $\sigma_\ell = -1$ and with velocity $\bar{v}(t^-_{j+\ell}) = v_{j+\ell} - \omega_\ell(\omega_\ell \cdot (\bar{v}(t_{k_\ell}) - v_{j+\ell}))$, if $\sigma_\ell = 1$. In the interval $(t_\ell, t_{\ell+1})$ the particle evolve according to $T^{j+\ell}_{-t}$, thus there are no internal collisions to be taken into account.

Hence, the main differences between $\bar{Z}(0)$ and $\tilde{Z}^\varepsilon(0)$ are due to the fact that the collisions occur at the same point instead than at distance ε and that there are no internal collisions between creations. The first difference is treated by using the continuity of the probability densities.

The absence of internal collisions in $\bar{Z}(0)$ is related to the problem of recollisions that requires a detailed analysis, and present the major novelty with respect to the case without forces. Before doing this, however, we want to show that the two sums (3.20) and (3.23) can be bounded by absolutely convergent series.

4. **Convergence of graph expansions.** For any sequence of functions $\mathbf{f} = \{f_j(Z_j)\}_{j\in\mathbb{N}}$ and any $\beta > 0$ we define the norm

$$\|f_j\|_{\beta,j} = \sup_{Z_j\in\Gamma_e^j} f_j(Z_j)\exp\left[\beta\sum_{i=1}^{j}\{\frac{1}{2}|v_i|^2 + U(x_i)\}\right]. \tag{4.1}$$

We need assumption on the initial distributions: We assume that there is $\beta > 0$ and $\xi > 0$ such that

$$\|P^{(j)}_{N,\varepsilon}(0)\|_{\beta,j} < \xi^j, \tag{4.2}$$

and

$$\|f_j(0)\|_{\beta,j} < \xi^j, \tag{4.3}$$

We prove the following

Proposition 4.1. *Suppose that* $\{f_j(0)\}_{j\in\mathbb{N}}$ *and* $\{P^{(j)}_{N,\varepsilon}(0)\}_{j\in\mathbb{N}}$ *satisfy (4.2) and (4.3) for some* β *and* ξ. *Then, for any* $\beta' < \beta$ *there is* t^* *and* $\mathcal{C}$ *such that for* $t < t^*$,

$$\sum_{n\geq 0}\|\mathcal{I}^{j,n}_{k,\varepsilon}\|_{\beta',j}(t) \leq \sum_{n\geq 0} c_n, \tag{4.4}$$

$$\sum_{n\geq 0}\|\mathcal{I}^{j,n}(t)\|_{\beta',j} \leq \sum_{n\geq 0} c_n, \tag{4.5}$$

with the series $\sum_{n\geq 0} c_n$ *geometrically convergent.*

Proof. We note that, by the conservation of energy in the evolution $T^{j,\varepsilon}_t$,

$$\|S^{j,\varepsilon}(t)P^{(j)}_{N,\varepsilon}\|_\beta \leq \|P^{(j)}_{N,\varepsilon}\|_\beta. \tag{4.6}$$

Moreover, for any $\beta' < \beta$, we claim

$$\|\sum_{i=1}^{j} C^{(j+1),i,\sigma}_{N,\varepsilon}P^{(j+1)}_{N,\varepsilon}\|_{\beta',j} \leq C\lambda^{-1}\beta^{-d/2}\mathrm{e}^{\beta B + (\beta-\beta')jB}$$
$$\|P^{(j+1)}_{N,\varepsilon}\|_{\beta,j+1}\left[\sqrt{\frac{j}{\beta-\beta'}} + \frac{j}{\sqrt{\beta}}\right]. \tag{4.7}$$

Indeed, from the definition of $C^{(j+1),i,\sigma}$, since $(N-j)\varepsilon^{d-1} < c\lambda^{-1}$ for N sufficiently large, we have,

$$\|\sum_{i=1}^{j} C^{(j+1),i,\sigma}_{N,\varepsilon}P^{(j+1)}_{N,\varepsilon}\|_{\beta',j} \leq \|P^{(j+1)}_{N,\varepsilon}\|_{\beta,j+1}\frac{C}{\lambda}\sum_{i=1}^{j}\int_{\mathbb{R}^d} dv_{j+1}$$
$$\int_{S^{d-1}_+} d\omega|(v_i - v_{j+1})\cdot\omega|\exp[-\beta\frac{1}{2}|v_{j+1}|^2 + U((x_i - \sigma\omega\varepsilon)]$$
$$\exp[-(\beta-\beta')\sum_{i=1}^{j}\frac{1}{2}|v_j|^2 + U((x_j)].$$

Since the potential energy is bounded from below, we can use the lower bound

$$\frac{1}{2}|v|^2 + U(x) > -B + \frac{|v|^2}{2}.$$

Moreover, $|(v_i - v_{j+1})\cdot\omega| \le |v_i| + |v_{j+1}|$. The following estimates are elementary:

$$\sum_{i=1}^{j} |v_i| \exp\left[(-\beta-\beta')\sum_{i=1}^{j}\frac{1}{2}|v_i|^2\right] \le c\sqrt{\frac{j}{\beta-\beta'}}, \tag{4.8}$$

$$\int_{\mathbb{R}^d} dv_{j+1}|v_{j+1}\exp\left[-\frac{\beta}{4}|v_{j+1}|^2\right] \le \frac{c}{\sqrt{\beta}}. \tag{4.9}$$

Combining them we get

$$\begin{aligned}&\|\sum_{i=1}^{j} C^{(j+1),i,\sigma}_{N,\varepsilon}P^{(j+1)}_{N,\varepsilon}\|_{\beta',j} \le \\ &C\beta^{-d/2}\frac{|S^{d-1}|}{2}\left[\sqrt{c\frac{j}{\beta-\beta'}}+c\frac{j}{\sqrt{\beta}}\right]\exp[\beta B + jB(\beta-\beta')]\|P^{(j+1)}_{N,\varepsilon}\|_{\beta,j+1}. \end{aligned}\tag{4.10}$$

By applying above estimate n times, with $\beta-\beta'$ replaced by $(\beta-\beta')/n$, and using the fact that the cardinality of the $\underline{\sigma}_n$ is 2^n, we obtain

$$\begin{aligned}&\|\mathcal{K}_{j,n}(t,t_1,\dots,t_n)\|_{\beta,j} \le \\ &\left(\frac{C\mathrm{e}^{\beta B}|S^{d-1}|}{\lambda\beta^{d/2}}\right)^n \prod_{\ell=1}^{n} \mathrm{e}^{\frac{\beta-\beta'}{n}(j+\ell-1)B}\left[\sqrt{\frac{j+\ell-1}{\beta-\beta'}}+\frac{j+\ell-1}{\sqrt{\beta}}\right]\|P^{(j+n)}_{N,\varepsilon}(0)\|_{\beta',j+n} \\ &\le \mathrm{e}^{\frac{\beta-\beta'}{n}jB}(j+n)^n\left(\frac{C\mathrm{e}^{2\beta B}|S^{d-1}|}{\lambda\beta^{d/2}}\right)^n\xi^{j+n}.\end{aligned}$$

Therefore

$$c_n \le (\mathcal{C}\xi)^j\frac{(j+n)^n}{n!}(\mathcal{C}\xi t)^n. \tag{4.11}$$

Since $\frac{(j+n)^n}{(j+n)!} \le C\mathrm{e}^n$ and $\frac{(j+n)!}{j!n!} \le 2^{j+n}$, the series $\sum_{n\ge 0} c_n$ is geometrically summable uniformly in ε and N, provided that $t < t_0 = (\mathcal{C}\xi\mathrm{e})^{-1}$, where $\mathcal{C}$ is a computable constant. The same argument can be applied to control the series (3.16), with exactly the same bound. □

Remark 4.1. The previous argument can be easily adapted to the cases where the force F does is not potential and/or if it depends smoothly on time. In those cases we do not have the conservation of energy, but clearly, the variation of kinetic energy in an interval of time of length $t < T$ is bounded by $T\sup_{x,t}|F(x,t)| = D_T$. Therefore it is enough to replace the constant B in (4.10) by the constant D_T, and then the time t^* is chosen to be the minimum between T and t_0.

Remark 4.2. In particular Proposition 4.1 shows that the series (3.14) is a local in time solution to the Boltzmann hierarchy (3.13). By the same arguments it can also be proved to that it is unique.

Remark 4.3. Note that the presence of the force only changes the value of the constant $\mathcal{C}$ because of the contribution due to the negative part of the potential energy, which is bounded by B.

5. **Recollision and term by term convergence.** A more substantial modification to the Lanford argument is made necessary by the presence of the force in the term by term convergence.

To discuss the convergence of the sum (2.27) to the series (3.14) we need an assumption on the convergence of the initial conditions. We assume that, for each $j > 0$ the functions $P^{(j)}_{N,\varepsilon}(0)$ are continuous and

$$P^{(j)}_{N,\varepsilon}(0) \to f_j(0) \tag{5.1}$$

uniformly in the compact sets of $(\Lambda \times \mathbb{R}^d)^j$ in the Boltzmann-Grad limit (BG), i.e. when $N \to \infty$ and $\varepsilon \to 0$ so that $N\varepsilon^{d-1} \to \lambda^{-1}$. A typical initial condition is

$$P^{(N)}_{N,\varepsilon}(Z_N) = \frac{1}{\mathcal{N}_\varepsilon} \prod_{i=1}^{N} f(z_i, 0) \prod_{k<i} \chi^\varepsilon(x_i, x_k), \tag{5.2}$$

where $f(z, 0)$ is any probability density on $\Lambda \times \mathbb{R}^d$, $\chi^\varepsilon(x, y)$ is the (smoothed) characteristic function of the set $\{(x, y) \,|\, |x-y| > \varepsilon\}$ and $\mathcal{N}_\varepsilon$ is the normalization factor. If $f(z, 0)$ is continuous, clearly $P^{(j)}_{N,\varepsilon}(0) \to \prod_{i=1}^{j} f(z_i, 0)$ uniformly on the compact sets.

We can prove the following

Proposition 5.1. *Suppose that for any $j > 0$ the functions $P^{(j)}_{N,\varepsilon}(0)$ are continuos and $P^{(j)}_{N,\varepsilon}(0) \to f_j(0)$ uniformly in the compact sets of $(\Lambda \times \mathbb{R}^d)^j$. Then there is a time $t^{**} > 0$, independent of n and j, such that $\mathcal{I}^{j,n}_{N,\varepsilon}(Z_j, t)$ converges to $\mathcal{I}^{j,n}(Z_j, t)$ for almost all Z_j and for $t < t^{**}$.*

Proof. By the results in Proposition 4.1, in order to prove the convergence of the sum (2.27) to the series (3.14) it is enough to show that for each fixed j and n, $\mathcal{I}^{j,n}_{N,\varepsilon}$ converges to $\mathcal{I}_{j,n}$ in the BG limit. By dominated convergence it is enough to show that, for each fixed $(t_1, \dots, t_n)$, $\mathcal{K}^{j,n}_{N,\varepsilon}(t, t_1, \dots, t_n)$ converges to $\mathcal{K}^{j,n}(t, t_1, \dots, t_n)$. By the decay properties in the velocity space which follow from Proposition 4.1, it is enough to show it for the contribution of each graph in a compact set $\mathcal{B}$ in the velocity space. Moreover, $\alpha_n(j) \to \lambda^{-n}$, so we need to show that $P^{(j+n)}_{N,\varepsilon}(\tilde{Z}^\varepsilon(0), 0)$ converges to $f_{(j+n)}(\bar{Z}^\varepsilon(0), 0)$ in a subset of $(\Lambda \times \mathcal{B})^{j+n}$ whose complement has a vanishing Lebesgue measure in the $\varepsilon \to 0$ limit. We stress that in this argument j and n are fixed, and the only parameter we are moving is ε.

We fix a tree $G(j, n)$. Associated to the tree there are the trajectories $\tilde{Z}^\varepsilon(s)$ given by $(\tilde{x}_1(s), \tilde{v}_1(s), \dots, \tilde{x}_{j+\ell}(s), \tilde{v}_{j+\ell}(s))$ for $s \in (0, t)$ and $\ell = 0, \dots, n$ such that $s \in (t_{\ell+1}, t_\ell]$. We introduce the characteristic function $\chi_{\rm rec}$ of the set of *recolliding* particles:

$$\chi_{\rm rec}(Z_j, \mathbf{t}_n, \underline{\omega}_n, \mathbf{v}_{j,n}) = \begin{cases} 1 \text{ if } \exists s \in [0, t], k \neq l \in \{1, \dots, j+n\}, |\tilde{x}_k(s) - \tilde{x}_l(s)| < \varepsilon \\ 0 \text{ otherwise.} \end{cases}$$

Without force, for any fixed $t > 0$, the set where $\chi_{\rm rec} = 1$ has clearly Lebesgue measure $\to 0$ as $\varepsilon \to 0$. In fact, in order to have a recollision between two specified particles, one has to choose the velocities in some very carefully specified sets whose measure goes to 0 as $\varepsilon \to 0$. This is enough to conclude the proof, since in this case, clearly $\tilde{Z}^\varepsilon(0) \to \bar{Z}(0))$ on the complement of the set of recolliding configuration, the functions $P_{N,\varepsilon}^{(j)}(Z_j,0)$ are continuous and converge to $f_j(Z_j,0)$ by the assumption (5.1).

With an external force, in general this is not true. The novelty is the following: suppose that at time $\bar{t}$ two specified particles are at distance ε and undergo a non grazing collision with incoming velocities v_1 and v_2. Then, going back in time, the straight trajectories without force would separate and this two particles would never collide again, if first some other collision occurs. The presence of the force changes this because the trajectories are no more straight and we cannot exclude that these two particles will meet again. The constant force provides a remarkable exception to this is, because in this case the two particles have the same acceleration which cancels out, so they do not meet again before another collision. If the force is not constant, then the accelerations of the two particles are different and at some later time they can get closer than ε. How long it will take depends on the projection of the relative velocities on the direction of the impact parameter ω and on the Lipshitz constant of the force.

Summarizing, in the presence of an external force, there are two kinds of recollisions. The first ones, similar to those without force, are due to very special choices of the velocities, which induces collisions of two particles who possibly never meet before, but both meet another specified particle. Thus, at least three particles are necessary to make such a recollision.

The second ones are consequence of the external force and correspond to the fact that after two particle collide, instead of separating, get again closer than ε because they are accelerated by slightly different forces. This kind of recollision occur even with only two particles and do not require any special choice of the velocities.

About the first type of collisions, they are dealt with as in the case without force. Indeed, since the force is uniformly Lipshitz continuous, it is easy to check that the set of velocities producing this kind of recollisions has vanishing measure as $\varepsilon \to 0$.

The second kind of recollisions cannot be excluded by small measure arguments. Indeed, as discussed in Remark 5.2, in some cases they occur with probability 1.

In order to treat them, we prove the following

Theorem 5.1. *Suppose that $x \to -\nabla U(x)$ is a bounded uniformly Lipshitz continuous function from $\mathbb{R}^d$ into $\mathbb{R}^d$ and $X(t,x,v)$ the solution to the equation*

$$\ddot{X} = -\nabla U(X), \quad X(0) = x, \quad \dot{X}(0) = v.$$

Let x_1, x_2, v_1, v_2 such that

- $|x_1 - x_2| = \varepsilon$;
- $|v_1 - v_2| \geq \varepsilon^\gamma$ *for some* $\gamma \in (0,1]$;
- $\frac{(x_1-x_2)}{|x_1-x_2|} \cdot \frac{(v_1-v_2)}{|v_1-v_2|} = \cos\alpha \geq \varepsilon^{1-\gamma}$.

Then, if ε is sufficiently small, there is $\mathcal{T} > 0$ independent of ε such that

$$|X(t,x_1,v_1) - X(t,x_2,v_2)| > \varepsilon$$

for any $t \in (0,\mathcal{T})$.

Proof. We have

$$X(x,v,t) = x + vt - \int_0^t ds \int_0^s du \nabla U(X(u,x,v)).$$

Let $\mathcal{L}(t) = x_1 - x_2 + (v_1 - v_2)t$ and

$$A = \{s \geq 0 : |X(s,x_1,v) - X(s,x_2,v_2) - \mathcal{L}(s)| \leq \frac{1}{2} s|\mathcal{L}(s)|\}.$$

The set A contains 0 because $X(0,x_1,v_1) - X(0,x_2,v_2) - \mathcal{L}(0) = 0$. Moreover $|X(s,x_1,v) - X(s,x_2,v_2) - \mathcal{L}(s)| = \mathcal{O}(s^2)$. Indeed

$$X(s,x_1,v) - X(s,x_2,v_2) - \mathcal{L}(s) = -\int_0^s d\tau \int_0^\tau du(\nabla U(X(u,x_1,v_1)) - \nabla U(X(u,x_2,v_2))).$$

Therefore

$$|X(s,x_1,v) - X(s,x_2,v_2) - \mathcal{L}(s)| \le 2\|\nabla U\|_\infty \frac{s^2}{2}.$$

On the other hand, setting $n = \dfrac{x_1 - x_2}{|x_1 - x_2|}$, we have

$$\mathcal{L}(s) = n|x_1 - x_2| + sn \otimes n \cdot (v_1 - v_2) + s(\mathbb{I} - n \otimes n) \cdot (v_1 - v_2).$$

By using the definition of $\cos\alpha$,

$$\begin{aligned}|\mathcal{L}(s)|^2 &= (|x_1 - x_2| + s\cos\alpha|v_1 - v_2|)^2 + s^2|v_1 - v_2|^2 \sin^2\alpha \\ &= |x_1 - x_2|^2 + s^2|v_1 - v_2|^2 + 2s\cos\alpha|x_1 - x_2||v_1 - v_2|\end{aligned}$$

which implies, by the conditions on x_i and v_i,

$$|\mathcal{L}(s)|^2 \ge \varepsilon^2(1 + 2s\varepsilon^{\gamma-1}\cos\alpha + \varepsilon^{2(\gamma-1)}s^2) \ge \varepsilon^2(1+2s)^2 \tag{5.3}$$

for ε sufficiently small. Therefore we have

$$\frac{1}{2}s|\mathcal{L}(s)| \ge \frac{1}{2}s\varepsilon(1+2s) = \mathcal{O}(s).$$

By continuity, A contains an interval of times s starting with $s = 0$. Let $\mathcal{T} > 0$ be the infimum of the times such that the condition defining A is violated. Previous argument shows that $\mathcal{T}$ is larger than 0. We need to prove that there is a lower bound for $\mathcal{T}$ independent of ε. By continuity,

$$|X(\mathcal{T},x_1,v_1) - X(\mathcal{T},x_2,v_2) - \mathcal{L}(\mathcal{T})| = \frac{1}{2}\mathcal{T}|\ell(\mathcal{T})|. \tag{5.4}$$

We now claim that there is a constant C independent of ε such that

$$|X(\mathcal{T},x_1,v_1) - X(\mathcal{T},x_2,v_2) - \mathcal{L}(\mathcal{T})| \le C|\mathcal{L}(\mathcal{T})|\mathcal{T}^2. \tag{5.5}$$

This, combined with (5.4) implies

$$C|\mathcal{L}(\mathcal{T})|\mathcal{T}^2 \ge \frac{1}{2}\mathcal{T}|\mathcal{L}(\mathcal{T})|,$$

and hence

$$\mathcal{T} \ge \frac{1}{2C}.$$

Proof of the claim (5.5): We note that for any $t \le \mathcal{T}$

$$|X(t,x_1,v_1) - X(t,x_2,v_2) - \mathcal{L}(t)| = \left| \int_0^t ds \int_0^s d\tau (\nabla U(X(\tau,x_1,v_2)) - \nabla U(X(\tau,x_2,v_2))) \right|$$

$$\le \|\nabla^2 U\|_\infty \int_0^t ds \int_0^s d\tau |X(\tau,x_1,v_1) - X(\tau,x_2,v_2) - \mathcal{L}(\tau)| + \|\nabla^2 U\|_\infty \int_0^t ds \int_0^s d\tau |\mathcal{L}(\tau)|$$

and (5.5) follows by using the Gronwall lemma and monotonicity of $|\mathcal{L}(s)|$. The constant C does not depend on ε, but only on the Lipshitz modulus of F.

Therefore, for $t \le \mathcal{T}$ we have

$$|X(t,x_1,v_1) - X(t,x_2,v_2)| \ge |\mathcal{L}(t)|(1 - \frac{t}{2}).$$

By (5.3),

$$|\mathcal{L}(t)|(1-\frac{t}{2}) \geq \varepsilon(1+2t)(1-\frac{t}{2}) = \varepsilon(1+\frac{3}{2}t-t^2) > \varepsilon,$$

provided that $0 < t < \frac{3}{2}$.

Hence, the two particles are not closer than ε on a time $t^{**} = \min\{\mathcal{T}, \frac{3}{2}\}$. □

To use this theorem we need to remove a larger set than $\{\chi_{\rm rec} = 0\}$. Indeed we need to add that at time t and before each of the collision times t_ℓ, $\ell = 1, \dots, n$ the conditions to apply the theorem are fulfilled. Therefore, we fix a $\gamma \in (0,1)$ and introduce the characteristic function $\chi_{\rm force}$, defined as follows:

$$\chi_{\rm force}(Z_j, \mathbf{t}_n, \underline{\omega}_n, \mathbf{v}_{j,n}) = \begin{cases} 1 & \text{if Conditions A is satisfied,} \\ 0 & \text{otherwise.} \end{cases} \tag{5.6}$$

Condition A

- *If* $\tilde{x}_{k_\ell}(t_{k_\ell}) - x_{j+\ell} = \varepsilon\omega_{k_\ell}$, *then* $|\tilde{v}_{k_\ell}(t_{k_\ell}) - v_{j+\ell}| > \varepsilon^\gamma$ *for each* $\ell = 1, \dots, n$*;*

$$-\omega_{k_\ell} \cdot \frac{\tilde{v}_{k_\ell}(t_{k_\ell}) - v_{j+\ell}}{|\tilde{v}_{k_\ell}(t_{k_\ell}) - v_{j+\ell}|} = \cos\alpha_{k_\ell} \geq \varepsilon^{1-\gamma}$$

- *if* $x_i - x_k = \varepsilon\omega_{i,k}$. *for* $i \neq k \in \{1, \dots, j\}$, *then* $|v_i - v_k| > \varepsilon^\gamma$*;*

$$-\omega_{i,k} \cdot \frac{v_i - v_k}{|v_i - v_k|} \cos\alpha_{k,l} \geq \varepsilon^{1-\gamma}.$$

Note that Theorem 5.1 is stated for clarity forward in time, but has to be applied here backward in time and hence we need to reverse the sign of $\cos\alpha$. Cleary, it still holds if we replace outgoing velocities with incoming velocities as we have done in Condition A. Theorem 5.1 ensures that in the set $\{\chi_{\rm force}(1 - \chi_{\rm rec} = 1)\}$ the convergence $\tilde{Z}^\varepsilon(0) \to \bar{Z}^\varepsilon(0))$ is true as long as $t < \min\{\mathcal{T}, \frac{3}{2}\}$. On the other hand, the complement of this set has vanishing Lebesgue measure as $\varepsilon \to 0$. Indeed, the Lebesgue measure of the set where Condition A is not satisfied is clearly small as $\varepsilon \to 0$, provided that $\gamma < 1$. □

Remark 5.1. The above argument would still hold if the force depends smoothly on time and/or is not derived by a potential.

Remark 5.2. The term by term convergence without force holds for any time, so the restriction of short times only depends on the applicability of Proposition 4.1. In the presence of an external force, one cannot expect to show the term by term convergence for any time with a general force. The argument used to prove Theorem 5.1 clearly hold for any time if the force is constant, since, in this case $X(s, x_1, v_1) - X(s, x_2, v_2) = \mathcal{L}(s) > \varepsilon^2$ for any $s < t$. On the other hand, in the presence of an elastic force with potential $V(x) = \frac{1}{2}k|x|^2$, the statement of Theorem 5.1 is false for times larger than $\frac{2\pi}{\sqrt{k}}$. Indeed, it is enough to consider the graph $G(1,1)$ with just an initial particle and a new born at time t_1. At time t_1 the two particles are at distance ε and undergo the prescribed collision. Then, since the evolution of $X(s, x_1, v_1) - X(s, x_2, v_2)$ is periodic, independently of the initial conditions, the particles will be again at distance ε at the time $t_1 - \frac{2\pi}{\sqrt{k}}$. Therefore, according to the evolution $T_t^{2,\varepsilon}$ they will collide again, while in the evolution T_t^2 there is no such collision.

The combination of Proposition 4.1 and Proposition 5.1 completes the proof of the validity of the Boltzmann equation with an external force, stated informally in the Introduction and precisely below.

Theorem 5.2. *Assume that for all the $j > 0$ the initial j-particle distributions $P^{(j)}_{N,\varepsilon}(Z_j, 0)$ are continuous and converge, uniformly on the compact sets of $\Lambda \times \mathbb{R}^d$ to $f_j(Z_j, 0)$ and there are $\beta > 0$ and $\xi > 0$ such that the inequalities*

$$\|P^{(j)}_{N,\varepsilon}(Z_j, 0)\|_{\beta,j} \le \xi^j, \quad \|f_j(Z_j, 0)\|_{\beta,j} \le \xi^j.$$

Then there is a time $t_* > 0$ such that, for any $t < t_*$, $P^{(j)}_{N,\varepsilon}(Z_j, t)$ solution to (2.26) converges almost everywhere in the sense of the Lebesgue measure on $\Lambda \times \mathbb{R}^d$ to $f_j(Z_j, t)$ solution to (3.13).

The following corollary establishes the validity of the Boltzmann equations and the *propagation of the chaos* for short times:

Corollary 5.1. *If the initial j-particle densities are given by (5.2), with $f(z, 0)$ such that $\|f(0)\|_{\beta,1}$ is bounded for some $\beta > 0$, then there is $t_* > 0$ such that, for $0 < t < t_*$, $P^{(j)}_{N,\varepsilon}(Z_j, t)$ converges almost everywhere to $\prod_{i=1}^{j} f(z_i, t)$ with $f(z, t)$ solution to the Boltzmann equation (3.4) with initial data $f(z, 0)$.*

Proof. This follows from Theorem 5.2 together with the observation that $\prod_{i=1}^{j} f(z_i, t)$ is solution to the (3.13) and is the unique by Remark 4.2. □

Remark 5.3. The combination of Remark 4.1 and Remark 5.1 implies that Theorem 5.2 and Corollary 5.1 still hold in the presence of a generic force $F(x, t)$ which is Lipshitz continuous both in x and t.

Acknowledgements. We wish to thank Laurent Desvillettes and Mario Pulvirenti for useful discussion. R. E. and R. M. also thank the kind hospitality of the Math. Department of the Beijing University and the Newton Institute for Mathematical Sciences in Cambridge, where part of this work was done. Y. Guo is supported in part by DMS 095255, a NSF FRG grant as well as a Chinese NSF grant. R. Esposito and R. Marra are partially supported by INDAM-GNFM and MIUR.

REFERENCES

[1] L. Arkeryd, R. Esposito, R. Marra, A. Nouri *Stability for Rayleigh-Benard convective solutions of the Boltzmann equation*, Arch. Rat. Mech. Anal **198**, 125–187 (2010)

[2] S. Bastea, R. Esposito, J. L. Lebowitz and R. Marra, *Binary Fluids with Long Range Segregating Interaction I: Derivation of Kinetic and Hydrodynamic Equation*, J. Statist. Phys, **101**, (2000), 1087.

[3] A. V. Bobylev, Alex Hansen, J. Piasecki and E. H. Hauge, *From the Liouville Equation to the Generalized Boltzmann Equation for Magnetotransport in the 2D Lorentz Model*, Jour. Statist. Phys., **102**, pages 1133-1150 (2001)

[4] E. A. Carlen, M. Carvalho , R. Esposito, J. L. Lebowitz and R. Marra, *Free Energy Minimizers for a Two–Species Model with Segregation and Liquid-Vapor Transition*, Nonlinearity **16**, 1075–1105 (2003).

[5] C. Cercignani, R. Illner and M. Pulvirenti, The Mathematical Theory of Dilute Gases, Springer-Verlag (1994)

[6] R. Duan, T. Yang and C. Zhu, *Global Existence to Boltzmann Equation with External Force in Infinite Vacuum*, Journal of Mathematical Physics 46, 053307 (2005).

[7] R. Duan, S. Ukai, T. Yang, H. Zhao, *Optimal Decay Estimates on the Linearized Boltzmann Equation with Time Dependent Force and their Applications*, Communications in Mathematical Physics, 277, no. 1, (2008).

[8] R. Esposito, Y. Guo and R. Marra, *Phase Transition in a Vlasov-Boltzmann Binary Mixture*, Commun. Math. Phys. **296**, pages 1–33 (2010)

[9] R. Esposito, R. Marra, J. L. Lebowitz, *Solutions to the Boltzmann Equation in the Boussinesq Regime*, J. Stat. Phys. **90**, 1129-1178 (1998).

[10] Y. Guo, *The Vlasov-Poisson-Boltzmann system near vacuum,* Comm. Math. Phys. 218 (2001), no. 2, 293-313.

[11] Y. Guo,*The Vlasov-Poisson-Boltzmann system near Maxwellians,* Comm. Pure Appl. Math. LV, 1104-1135 (2002).

[12] Y. Guo, *The Vlasov-Boltzmann-Maxwell system near Maxwellians,* Invent. Math., 153, 593-630 (2003).

[13] O. E. Lanford III, *The evolution of Large Classical systems*, in "Dynamical systems, theory and Applications" , J. Moser ed., Lecture Notes in Physics **35**. pp. 1–111, Springer Berlin (1975)

[14] S. Ukai, T. Yang and H. Zhao, *Global Solutions to the Boltzmann Equation with External Forces*, Analysis and Applications (Singapore) 3, no. 2, 157–193 (2005)

Received September 2006; revised February 2007.

E-mail address: esposito@univaq.it
E-mail address: guoy@dam.brown.edu
E-mail address: marra@roma2.infn.it