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Displacement convexity and minimal fronts at phase boundaries[1]

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Abstract

We show that certain free energy functionals that are not convex with respect to the usual convex structure on their domain of definition, are strictly convex in the sense of displacement convexity under a natural change of variables. We use this to show that in certain cases, the only critical points of these functionals are minimizers. This builds on previous work by Alberti and Bellettini who first found such an alternate convex structure. The development here in terms of displacement convexity permits us to treat multicomponent systems, and provides two new examples of displacement convex functionals, and one that it jointly displacement convex in the multicomponent setting.

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1. Introduction

1.1. The variational problem

We consider minimization problems for a type of functional that arises in the study of phase segregation in statistical mechanical systems. Let $F(m)$ be a function on the real line that is continuous and strictly positive except at $m = a$ and $m = b$ with $a < b$. A good example to bear in mind is the "double well potential"

$$F(m) = \frac{1}{4}(m^2 - 1)^2 \ ,$$

where of course $a = -1$ and $b = 1$.

Let $\mathcal{C}_{a,b}$ be the set of functions $m(x)$ from $\mathbb{R}$ to $\mathbb{R}$ such that

$$\lim_{x\to-\infty} m(x) = a \qquad \text{and} \qquad \lim_{x\to+\infty} m(x) = b \ .$$

The numbers a and b represent the values of the order parameter m in two phases of a statistical mechanical system. For example, $m = a$ might correspond to a vapor phase, and $m = b$ to a liquid phase.

Then a function $m(x)$ in $\mathcal{C}_{a,b}$ denotes a possible *transition profile* across the boundary segregating the two different phases. The actual profile that one would expect to see would be one that minimizes the free energy cost of making such a transition. Here, the free energy functional $\mathcal{F}$ to be minimized on $\mathcal{C}_{a,b}$ is

$$\mathcal{F}(m) = \int_{\mathbb{R}} F(m(x))\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} (m(x) - m(y))^2 J(x-y)\mathrm{d}x\mathrm{d}y \ , \qquad (1.1)$$

where $J(x)$ is a non-negative integrable function on $\mathbb{R}$. Let $\widehat{J}$ denote the total mass of J:

$$\widehat{J} = \int_{\mathbb{R}} J(x)\mathrm{d}x \ . \qquad (1.2)$$

The term $\int_{\mathbb{R}} F(m(x))\mathrm{d}x$ is due to short range interactions and entropy effects, while the term $\int_{\mathbb{R}}\int_{\mathbb{R}}(m(x)-m(y))^2 J(x-y)\mathrm{d}x\mathrm{d}y$ is due to long range interactions. This long range term in the free energy suppresses sharp transitions, as does the gradient term in the familiar but purely phenomenological Van der Waals model [12]. For more discussion of the physical context of the problem, see [7].

Much useful information can be deduced from the specific form of the minimizing profiles. In particular, the surface tension at a phase boundary is the minimum value of $\mathcal{F}(m)$ on $\mathcal{C}_{a,b}$; see [3] for more information. Hence we ask:

• *What is the minimum value of $\mathcal{F}(m)$ as m ranges over $\mathcal{C}_{a,b}$, and are the minimizing profiles, if any, unique up to translation?*

Actually, the existence of minimizers is relatively simple to prove using the rearrangement inequalities to be discussed below. However, because of the translation invariance, they are never unique: Any translate of a minimizer is again a minimizer. It is less simple to show that this is the only degeneracy.

1.2. Displacement convexity and uniqueness of fronts

For a particular choice of F in the free energy functional specified in (1.1), this problem has been solved in a series of papers [9],[10] by De Masi, Orlandi, Triolo and Presutti, building on previous unpublished work of Dal

Passo and de Mottoni [8] Their solution involves the construction of a dynamics that is dissipative for the free energy functional, and then a careful analysis of limits along the time evolution for this dynamics.

Another approach that we further develop here has been introduced by Alberti and Bellettini [2], [1]. They discovered an alternative convex structure which renders the variational problem for (1.1) convex, and used this to study the existence problem in [2]. Later, Alberti [1] returned to the problem and proved a uniqueness result that affirmatively answers the question raised above for this one component model.

Our goal here is to treat certain two component systems. Motivated by this problem, we were led to reconsider the single component problem from the point of view of McCann's notion of *displacement convexity* [11]. In fact, the minimization problem for (1.1) is challenging largely because the functional $\mathcal{F}$ is *not convex* on $\mathcal{C}_{a,b}$ in the usual way: For $0 < \lambda < 1$, and m_0 and m_1 in $\mathcal{C}_{a,b}$, define $m_\lambda = (1-\lambda)m_0 + \lambda m_1$ and note that $m_\lambda \in \mathcal{C}_{a,b}$. However, due to the non convexity of the potential function F, it is *not* true in general that $\mathcal{F}(m_\lambda) \leq (1-\lambda)\mathcal{F}(m_0) + \lambda\mathcal{F}(m_0)$.

In [11], McCann, building on groundbreaking work of Brenier [5], introduced an alternative convex structure on the space of probability densities on $\mathbb{R}^n$, and used this to prove existence and uniqueness results for minimizers of functionals that were not convex in the usual sense. We shall show here that the minimization problem for (1.1), as well as for a two component model of this type, can be handled within this framework. In the process, we provide two new examples of strictly displacement convex functionals, the second of which is jointly displacement convex. It turns out that the alternative convex structure introduced in [2] is equivalent to the displacement convexity in this one dimensional setting, although the approach is quite different. We shall see that developing the alternative convex structure explicitly in terms of displacement convexity has advantages, especially for the two component system, when one seeks to prove a uniqueness result.

We now describe the alternate convex structure with respect to which $\mathcal{F}$ *is* convex. This second convex structure cannot be defined on all of $\mathcal{C}_{a,b}$, but only on the subset of *monotone* profiles $\mathcal{M}_{a,b}$. Nothing is lost in this restriction, as rearrangement inequalities show that minimizers of $\mathcal{F}$ on $\mathcal{C}_{a,b}$ must actually lie in $\mathcal{M}_{a,b}$; see [1] and Theorem 6.1 below.

Now, any right–continuous profile $m(x)$ in $\mathcal{M}_{a,b}$ can be written in the form

$$m(x) = a + (b-a)\int_{(-\infty,x)} \mathrm{d}\mu(y)$$

where μ is a uniquely determined probability measure on $\mathbb{R}$. This identification of $\mathcal{M}_{a,b}$ and the set of probability measures on $\mathbb{R}$ allows us to look at $\mathcal{F}$ as a functional defined on probability measures.

This is a useful perspective since there is an alternative convex structure on the set of probability measures on $\mathbb{R}$ (or more general domains) that was introduced by McCann, and which we describe below. A functional on probability measures is said to be *displacement convex* if it is convex with respect to this alternative structure. We shall show here that $\mathcal{F}$, regarded as a functional on probability measures is, in fact, displacement convex. Using this, we shall show that any solution in $\mathcal{M}_{a,b}$ of the Euler–Lagrange equation

$$m(x) = \frac{1}{\widehat{J}}\left(\int_{\mathbb{R}} J(x-y)m(y)\mathrm{d}y - F'(m(x))\right) \tag{1.3}$$

for our variational problem concerning (1.1) is in fact a minimizer. Solutions to (1.3) can easily be constructed by iteration, and thus from here, surface tensions may be readily computed.

This solution to the variational problem has the advantage of applying also for free energy functionals in certain multicomponent systems, in which the determination of the minimizers has not been previously treated. Indeed, our motivation was to be able to rigorously determine the surface tension in such systems. However, we shall first present our simple solution of the minimization problem for the single component free energy functional $\mathcal{F}$ specified in (1.1), and only after this is complete, shall we treat the multicomponent case.

2. The alternative convex structure

2.1. The reduction to monotone profiles

First of all, notice that if we seek to minimize $\mathcal{F}$ on $\mathcal{C}_{a,b}$, we need only consider profiles m for which $a \leq m(x) \leq b$ for all x. Indeed, for any $m \in \mathcal{C}_{a,b}$, define $\widehat{m}$ by

$$\widehat{m}(x) = \min\{b, \max\{a, m(x)\}\} \ .$$

Then it is clear that $\mathcal{F}(\widehat{m}) \leq \mathcal{F}(m)$ with equality only in case $\widehat{m} = m$.

We now recall a notion of rearrangement due to Alberti [1]. For any Borel measurable set A, let $|A|$ denote its Lebesgue measure. The rearrangement is defined for Borel sets $A \subset \mathbb{R}$ such that $|A\Delta(0,\infty)| < \infty$, where $A\Delta B = A\backslash B \cup B\backslash A$ is the symmetric difference of A and B. For such a set A, define the rearranged set A^* by

$$A^* = [a, \infty) \qquad \text{where} \qquad a = |(0,\infty)\backslash A| - |A\backslash(0,\infty)| \ .$$

Any function m in $\mathcal{C}_{a,b}$ that takes values in $[a,b]$ can be represented in "layer–cake" form:

$$m(x) = \int_a^b 1_{\{m>z\}}(x)\mathrm{d}z \ .$$

For each $z \in (a,b)$, the set $\{m > z\}$ certainly has the property that $|\{m > z\}\Delta(0,\infty)| < \infty$. Hence one can define the rearrangement of m itself through

$$m^*(x) = \int_a^b \left(1_{\{m>z\}}\right)^* (x)\mathrm{d}z \ .$$

Alberti shows that for any two such functions m_1 and m_2,

$$\int_{\mathbb{R}} |m_1^*(x) - m_2^*(x)|^2 \mathrm{d}x \le \int_{\mathbb{R}} |m_1(x) - m_2(x)|^2 \mathrm{d}x \ .$$

In particular, with m being any function in $\mathcal{C}_{a,b}$ that takes values in $[a,b]$, and h any real number, let $m_1(x) = m(x)$, and $m_2(x) = m(x+h)$. Then

$$\int_{\mathbb{R}} |m^*(x) - m^*(x+h)|^2 \mathrm{d}x \le \int_{\mathbb{R}} |m(x) - m(x+h)|^2 \mathrm{d}x \ ,$$

so that

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |m^*(x) - m^*(x+h)|^2 \mathrm{d}x \right) J(h)\mathrm{d}h \le \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |m(x) - m(x+h)|^2 \mathrm{d}x \right) J(h)\mathrm{d}h \ .$$

This of course means that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} (m^*(x) - m^*(y))^2 J(x-y)\mathrm{d}x\mathrm{d}y \le \int_{\mathbb{R}} \int_{\mathbb{R}} (m(x) - m(y))^2 J(x-y)\mathrm{d}x\mathrm{d}y \ . \tag{2.1}$$

In fact, Alberti shows (see Theorem 2.11 in [1]) that there is equality in (2.1) if and only if $m = m^*$.

Of course, $\int_{\mathbb{R}} F(m^*(x))\mathrm{d}x = \int_{\mathbb{R}} F(m(x))\mathrm{d}x$, and so we have $\mathcal{F}(m^*) \le \mathcal{F}(m)$ with equality if and only if $m = m^*$. Thus, we may restrict our search for minimizers to $\mathcal{C}_{a,b}$, the subset of monotone increasing profiles in $\mathcal{C}_{a,b}$.

2.2. Displacement convexity of $m \mapsto \int_{\mathbb{R}} F(m(x))\mathrm{d}x$

As we have explained in the introduction, if m is any profile in $\mathcal{M}_{a,b}$, then $(m(x)-a)/(b-a)$ is the cumulative distribution function of a uniquely determined probability measure μ:

$$\frac{m(x) - a}{b - a} = \int_{(-\infty,x)} \mathrm{d}\mu(y) \ .$$

For each m in $\mathcal{M}_{a,b}$, define $x(m)$ to be the inverse function: For $m \in (a,b)$,

$$x(m) = \sup\{ \ x \ : \ m(x) < m \ \} \ . \tag{2.2}$$

Then of course, $m(x)$ is the inverse function of $x(m)$, so that for x in $\mathbb{R}$,

$$m(x) = \sup\{ \ m \ : \ x(m) < x \ \} \ . \tag{2.3}$$

Clearly,

$$\int_{\mathbb{R}} F(m(x))\mathrm{d}x = \int_a^b F(m)\frac{\mathrm{d}x(m)}{\mathrm{d}m}\mathrm{d}m \ .$$

Let m_0 and m_1 be any two elements of $\mathcal{M}_{a,b}$, and let x_0 and x_1 denote their respective inverse functions. Then for any $\lambda \in (0,1)$, define $x_\lambda(m)$ by

$$x_\lambda(m) = (1-\lambda)x_0(m) + \lambda x_1(m)\ . \tag{2.4}$$

Note that x_λ is also the inverse function of an element of $\mathcal{M}_{a,b}$, which we shall call m_λ. That is,

$$m_\lambda(x) = \sup\{\ m\ :\ (1-\lambda)x_0(m) + \lambda x_1(m) < x\ \}\ . \tag{2.5}$$

Then,

$$\begin{aligned}\int_{\mathbb{R}} F(m_\lambda(x))\mathrm{d}x &= \int_a^b F(m)\frac{\mathrm{d}x_\lambda(m)}{\mathrm{d}m}\mathrm{d}m \\ &= (1-\lambda)\int_a^b F(m)\frac{\mathrm{d}x_0(m)}{\mathrm{d}m}\mathrm{d}m + \lambda\int_a^b F(m)\frac{\mathrm{d}x_1(m)}{\mathrm{d}m}\mathrm{d}m \\ &= (1-\lambda)\int_{\mathbb{R}} F(m_0(x))\mathrm{d}x + \lambda\int_{\mathbb{R}} F(m_1(x))\mathrm{d}x\ .\end{aligned} \tag{2.6}$$

This tells us that along the interpolation m_λ between m_0 and m_1 provided by (2.5), the function $\lambda \mapsto \int_{\mathbb{R}} F(m_\lambda(x))\mathrm{d}x$ is affine, and in particular, is convex. This is not the case for the standard interpolation given by

$$\widetilde{m}_\lambda(x) = (1-\lambda)m_0(x) + \lambda m_1(x)\ , \tag{2.7}$$

since

$$\lambda \mapsto \int_{\mathbb{R}} F(\widetilde{m}_\lambda(x))\mathrm{d}x$$

is *not*, in general, convex. That is, taking convex combinations in terms of the inverse function $x(m)$, as in (2.5), instead of $m(x)$ itself, as in (2.7), has "cured" the non-convexity of the functional $m \mapsto \int_{\mathbb{R}} F(m(x))\mathrm{d}x$.

Of course, this will only be useful if the functional

$$m \mapsto \int_{\mathbb{R}}\int_{\mathbb{R}} (m(x) - m(y))^2 J(x-y)\mathrm{d}x\mathrm{d}y\ , \tag{2.8}$$

which was convex in the usual way, is still convex with the new convex structure. This is not at all obvious, but the main result of the next section asserts that this is the case.

The approach of Alberti and Bellettini [2], which we discovered only after our work was complete, was to rewrite the interaction directly in terms of x_m, and to show that it is convex.

However, it turns out that the convex structure in (2.4) is something that is by now well–known; it is the *displacement convexity* structure introduced by McCann. Making this connection will facilitate showing the *strict* convexity of $m \mapsto \int_{\mathbb{R}}\int_{\mathbb{R}}(m(x) - m(y))^2 J(x-y)\mathrm{d}x\mathrm{d}y$ under this convex structure. This point was left open in [2], who explicitly asked whether one could extend the ideas to give a direct proof of uniqueness. Although Alberti [1] did later return to address the issue, we shall see here that the strict convexity is quite clear from the perspective of displacement convexity.

Displacement convexity is usually introduced as a convex structure in a set of probability measures. Given a probability measure μ_0 on $\mathbb{R}$, and a measurable map $T : \mathbb{R} \to \mathbb{R}$, we define the *push forward of* μ_0 *under* T, $T\#\mu_0$, by

$$\int_{\mathbb{R}} \phi(T(x)) \mathrm{d}\mu_0(x) = \int_{\mathbb{R}} \phi(y) \mathrm{d}(T\#\mu_0)(y) \ , \tag{2.9}$$

for all bounded, continuous functions ϕ.

Given two probability measures μ_0 and μ_1, it is natural to ask whether there is any *monotone* map T such that $T\#\mu_0 = \mu_1$. Suppose this is the case. Then for any $a \in R$, let ϕ_a be the step function

$$\phi_a(x) = 1_{(-\infty,a]}(x) \ .$$

Therefore,

$$\int_{\mathbb{R}} \phi_a(T(x)) \mathrm{d}\mu_0(x) = \int_{\mathbb{R}} \phi_a(y) \mathrm{d}\mu_1(y) \ ,$$

and hence

$$\int_{-\infty}^{T^{-1}(a)} \mathrm{d}\mu_0 = \int_{-\infty}^{a} \mathrm{d}\mu_1 \ . \tag{2.10}$$

Let m_0 and m_1 be the cumulative distribution functions of μ_0 and μ_1, respectively. Then (2.10) entails that $m_0(T^{-1}(a)) = m_1(a)$ for all a, or, what is the same thing

$$m_0(a) = m_1(T(a)) \tag{2.11}$$

for all a. In terms of the inverse functions $x_0(m)$ and $x_1(m)$, this means that

$$T(a) = x_1(m_0(a)) \tag{2.12}$$

and

$$x_1(m) = T(x_0(m)) \ . \tag{2.13}$$

As long as μ_0 and μ_1 have strictly positive densities, (2.12) does indeed define a monotone map T, and then it is very easy to see that with T defined by (2.12), $T\#\mu_0 = \mu_1$, and in fact, this is true without further technical hypotheses.

We now interpolate the map T, and hence the corresponding probability measures μ_0 and μ_1 and the corresponding cumulative distribution functions m_0 and m_1 as well. For all $\lambda \in [0,1]$, define T_λ by

$$T_\lambda(x) = (1-\lambda)x + \lambda T(x) \ . \tag{2.14}$$

If we define $x_\lambda(m)$ by

$$x_\lambda(m) = T_\lambda(x_0(m)) \ ,$$

then clearly x_λ is given by (2.4).

The displacement convex structure on probability measures on $\mathbb{R}$ is given by $\mu_\lambda = T_\lambda \# \mu_0$, and so it is nothing other than the convex structure (2.4), expressed in terms of probability measures instead of cumulative distribution functions. When μ_1 and μ_2 have strictly positive densities, so that T is given by (2.11), we denote the density of μ_λ by ρ_λ, and write

$$\rho_\lambda = T_\lambda \# \rho_1 \ . \tag{2.15}$$

We summarize the main result of this section in a theorem:

Theorem 2.1. *Let $\lambda \mapsto m_\lambda$ be the displacement interpolation between m_0 and m_1 in $\mathcal{M}$. Then for $0 \le \lambda \le 1$,*

$$\int_{\mathbb{R}} F(m_\lambda(x))\mathrm{d}x = (1-\lambda)\int_{\mathbb{R}} F(m_0(x))\mathrm{d}x + \lambda \int_{\mathbb{R}} F(m_1(x))\mathrm{d}x \ .$$

3. Displacement convexity of the interaction energy

Let $\mathcal{M}$ denote the class of cumulative distribution functions on $\mathbb{R}$. It will be convenient to reduce $\mathcal{M}_{a,b}$ to $\mathcal{M}$: Making the obvious change of variables, we can assume without loss of generality that $a = 0$ and $b = 1$. We do so.

Given any $m \in \mathcal{M}$, let μ be the corresponding probability measure, so that $m(x) = \int_{-\infty}^x \mathrm{d}\mu(y)$. The first step in the investigation of the interaction energy is to rewrite it as a functional of μ instead of m. This is done in the following lemma:

Lemma 3.1 *Assume that $\int_{\mathbb{R}} |s|J(s)\mathrm{d}s < \infty$. Define W in terms of J by setting*

$$W(u) = \int_u^\infty (s-u)(J(s)+J(-s))\mathrm{d}s \ . \tag{3.1}$$

for $u \ge 0$ and $W(u) = W(-u)$ for $u < 0$. Then

$$\int_{\mathbb{R}}\int_{\mathbb{R}} (m(x)-m(y))^2 J(x-y)\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}}\int_{\mathbb{R}} W(z-w)\mathrm{d}\mu(z)\mathrm{d}\mu(w) \ .$$

W is a symmetric function, and is convex on $(0,\infty)$ and on $(-\infty,0)$, though not on all of $\mathbb{R}$.

Proof: Since for $x < y$, $m(x) - m(y) = \int_x^y \mathrm{d}\mu(z)$, we have from the Fubini Theorem that

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} (m(x)-m(y))^2 J(x-y)\mathrm{d}x\mathrm{d}y = \\
&\int_{\mathbb{R}}\int_{\mathbb{R}} \left[\int_{\mathbb{R}}\int_{\mathbb{R}} 1_{[x,y]}(z)1_{[x,y]}(w)\mathrm{d}\mu(z)\mathrm{d}\mu(w)\right] J(x-y)\mathrm{d}x\mathrm{d}y = \\
&\int_{\mathbb{R}}\int_{\mathbb{R}} \left[\int_{\mathbb{R}}\int_{\mathbb{R}} 1_{[x,y]}(z)1_{[x,y]}(w)J(x-y)\mathrm{d}x\mathrm{d}y\right] \mathrm{d}\mu(z)\mathrm{d}\mu(w) \ .
\end{aligned} \tag{3.2}$$

Thus if we define $V(z-w)$ by

$$V(z-w) = \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{[x,y]}(z) 1_{[x,y]}(w) J(x-y) \mathrm{d}x \mathrm{d}y \ ,$$

we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}} (m(x) - m(y))^2 J(x-y) \mathrm{d}x \mathrm{d}y = \int_{\mathbb{R}} \int_{\mathbb{R}} V(z-w) \mathrm{d}\mu(z) \mathrm{d}\mu(w) \ .$$

We next show that $V = W$. To do this, write

$$J_+(x) = \begin{cases} J(x) & \text{for } x > 0 \ , \\ 0 & \text{for } x \leq 0 \end{cases} \qquad \text{and} \qquad J_-(x) = J(x) - J_+(x) \ .$$

We first consider $\int_{\mathbb{R}} \int_{\mathbb{R}} 1_{[x,y]}(z) 1_{[x,y]}(w) J_+(x-y) \mathrm{d}x \mathrm{d}y$. Make the change of variables $s = y - x$ and $t = (x+y)/2$. Then $\mathrm{d}x\mathrm{d}y = \mathrm{d}s\mathrm{d}t$, and

$$1_{[x,y]}(z) 1_{[x,y]}(w) = 1_{[t-s/2, t+s/2]}(z) 1_{[t-s/2, t+s/2]}(w) \ .$$

This quantity is zero unless

$$|z-w| \leq s \qquad \text{and} \qquad |2t - (x+w)| \leq s - |z-w| \ ,$$

in which case it is one. Therefore

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{R}} 1_{[x,y]}(z) 1_{[x,y]}(w) J_+(x-y) \mathrm{d}x \mathrm{d}y \\ &= \int_{|z-w|}^{\infty} \left(\int_{(z+w)/2 - (s-|z-w|)/2}^{(z+w)/2 + (s-|z-w|)/2} \mathrm{d}t \right) J_+(s) \mathrm{d}s \\ &= \int_{|z-w|}^{\infty} (s - |z-w|) J_+(s) \mathrm{d}s \ . \end{aligned} \tag{3.3}$$

Doing the same calculation for the part involving J_-, we obtain that $V = W$ where W is given by (3.1). Note that W is clearly symmetric. Also, for $u > 0$,

$$W'(u) = -\int_u^{\infty} (J(s) + J(-s)) \mathrm{d}s \ ,$$

and so

$$W''(u) = J(u) + J(-u) \ ,$$

which is non–negative. Thus, W is convex on $(0, \infty)$, and on $(-\infty, 0)$ by symmetry. However, it is not convex on the whole real line. Notice that $W(0) = \int_0^{\infty} s(J(s) + J(-s)) \mathrm{d}s > 0$, while $\lim_{u \to \pm\infty} W(u) = 0$. □

We now prove the main result of this section:

Theorem 3.2. *Let $\lambda \mapsto m_\lambda$ be the displacement interpolation between m_0 and m_1 in $\mathcal{M}$, as defined in (2.5). Then for $0 < \lambda < 1$,*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} (m_\lambda(x) - m_\lambda(y))^2 J(x-y)\mathrm{d}x\mathrm{d}y \leq$$
$$(1-\lambda)\int_{\mathbb{R}}\int_{\mathbb{R}} (m_0(x) - m_0(y))^2 J(x-y)\mathrm{d}x\mathrm{d}y \tag{3.4}$$
$$+ \lambda \int_{\mathbb{R}}\int_{\mathbb{R}} (m_1(x) - m_1(y))^2 J(x-y)\mathrm{d}x\mathrm{d}y \ . \tag{3.5}$$

If J is strictly positive on some interval, and m_0 has a strictly positive derivative almost everywhere, there is equality if and only if m_1 is a translate of m_0.

Proof: If W were convex on all of $\mathbb{R}$, the displacement convexity of the interaction energy would be a classical result of McCann [11]. However, in one dimension, the partial convexity of W that was established in Lemma 3.1 suffices. This is because the map T_λ is monotone for all λ. Therefore, if $z > w$, $T_\lambda(z) > T_\lambda(w)$ for all λ. Hence, as we vary λ, $T_\lambda(z) - T_\lambda(w)$ stays in a domain of convexity of W.

Therefore, if $\mathrm{d}\mu_\lambda = T_\lambda \# \mathrm{d}\mu_0$ is the displacement interpolation between $\mathrm{d}\mu_0$ and $\mathrm{d}\mu_1$, we have from (2.4) that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} W(z-w)\mathrm{d}\mu_\lambda(z)\mathrm{d}\mu_\lambda(w) = \int_{\mathbb{R}}\int_{\mathbb{R}} W(T_\lambda(z) - T_\lambda(w))\mathrm{d}\mu_0(z)\mathrm{d}\mu_0(w) \ . \tag{3.6}$$

Introduce the map $S(x)$ defined by $S(x) = T(x) - x$. Then,

$$T_\lambda(z) - T_\lambda(w) = [z-w] + \lambda[S(z) - S(w)] \ ,$$

and we can rewrite (3.6) as

$$\int_{\mathbb{R}}\int_{\mathbb{R}} W(z-w)\mathrm{d}\mu_\lambda(z)\mathrm{d}\mu_\lambda(w) = \int_{\mathbb{R}}\int_{\mathbb{R}} W([z-w]+\lambda[S(z)-S(w)])\mathrm{d}\mu_0(z)\mathrm{d}\mu_0(w) \ . \tag{3.7}$$

By the remarks made above, the right hand side is clearly a convex function of λ. In fact, under mild assumptions on μ_0 or J, it is strictly convex unless T is simply a translation.

To see this *formally*, let J be symmetric for simplicity of notation, and differentiate the right hand side of (3.7) twice in λ, finding

$$\int_{\mathbb{R}}\int_{\mathbb{R}} 2J\left([z-w] + \lambda[S(z) - S(w)]\right)[S(z) - S(w)]^2 \mathrm{d}\mu_0(z)\mathrm{d}\mu_0(w) \ .$$

If this vanishes for all λ, then

$$\int_{\mathbb{R}}\int_{\mathbb{R}} 2J(z-w)[S(z) - S(w)]^2 \mathrm{d}\mu_0(z)\mathrm{d}\mu_0(w) = 0 \ .$$

If J is strictly positive and if μ_0 has a strictly positive density, then this is possible if and only if S is constant, and that of course means that T is a translation.

To make this argument rigorous, and to relax the hypotheses, let $f(\lambda)$ denote the right hand side of (3.4) minus the left hand side. Then, with $g(z, w, \lambda)$ defined by

$$g(z, w, \lambda) = [\lambda W(z - w) + (1 - \lambda)W((z - w) + (S(z) - S(w)))]$$

$$-W((z - w) + \lambda(S(z) - S(w))) \ ,$$

we have

$$f(\lambda) = \int_{\mathbb{R}} \int_{\mathbb{R}} g(z, w, \lambda) \mathrm{d}\mu_0(z) \mathrm{d}\mu_0(w) \ .$$

Since the integrand is non negative, we have for any measurable subsets A and B of $\mathbb{R}$,

$$f(\lambda) \geq \int_A \int_B g(z, w, \lambda) \mathrm{d}\mu_0(z) \mathrm{d}\mu_0(w) \ . \tag{3.8}$$

Suppose that J is strictly positive on the open interval $I = (y_0 - \delta/2, y_0 + \delta/2)$. Then I is an interval of strict convexity of W, so that whenever $w - z \in I$, $\lambda \mapsto g(z, w, \lambda) > 0$ on $(0, 1)$ unless $S(z) = S(w)$. However, if S is not constant almost everywhere, we can find an arbitrarily small interval about some z_0 on which it has strictly positive oscillation. In particular, we can find a z_0 and an $\epsilon > 0$ so that $\int_{z_0-\delta/2}^{z_0+\delta/2} (S(z) - c)^2 \mathrm{d}z > \epsilon$ for all c. Let $A = (z_0 - \delta/2, z_0 + \delta/2)$, and let $B = (y_0 + x_0 - \delta/2, y_0 + x_0 + \delta/2)$. Then for all z in A and w in B, $z - w$ belongs to I. Moreover, for every w in B, $\int_A (S(z) - S(w))^2 \mathrm{d}w > 0$, so $|S(z) - S(w)| > 0$ on a subset of A of positive Lebesgue measure. Since μ_0 has a strictly positive density, this ensures that the right hand side of (3.8) is strictly positive. □

It is clear that the conditions on J and m_0 that are invoked to ensure strict convexity can be relaxed, though they are already quite general.

We close this section a remark. If the profile m is continuously differentiable with $m'(x) = \rho(x)$, and $\int_{\mathbb{R}} J(x)\mathrm{d}x = 1$, then

$$\lim_{h \to 0} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(m(x) - m(y))^2}{h^2} \frac{1}{h} J\left(\frac{x - y}{h}\right) \mathrm{d}x \mathrm{d}y = \int_{\mathbb{R}} \rho^2(x) \mathrm{d}x \ .$$

It is already well known that the functional $\rho \mapsto \int_{\mathbb{R}} \rho^2(x)\mathrm{d}x$ is displacement convex, so the fact that Theorem 3.2 gives another proof of this is not of great interest. However, the connection between the two functionals at least gives one suggestion as to why the interaction functional might be expected to be displacement convex.

4. For the functional $\mathcal{F}$, critical points are minimizers

We are now ready to prove the main theorem for $\mathcal{F}$:

Theorem 4.1. *If m_0 is any critical point of $\mathcal{F}$ in $\mathcal{M}$, and m is any other profile in $\mathcal{M}$, then*

$$\mathcal{F}(m) \geq \mathcal{F}(m_0)$$

and there is equality if and only if m is a translate of m_0.

Proof: Let m_λ be the displacement interpolation between m_0 and m. Then $\lambda \mapsto \mathcal{F}(m_\lambda)$ is convex, and the derivative is zero at $\lambda = 0$. Hence m_0 is a minimizer of $\mathcal{F}$, so that $\mathcal{F}(m_\lambda) \geq \mathcal{F}(m_0)$, and there is equality if and only if $\lambda \mapsto \mathcal{F}(m_\lambda)$ is constant. But in this case, the strict displacement convexity of $\mathcal{F}$ ensures that m is a translate of m_0. □

5. Fronts in a binary fluid model

We now turn to the study of the analogous problem for a binary fluid model. The binary fluid model has been investigated in [6] and [7], and we refer to those papers for details. Although the arguments apply to that setting in full generality, we discuss here only a special case for sake of brevity. For further information and a numerical investigation of the minimizing fronts, see [4].

In what follows, $m(x)$ and $n(x)$ represent the particle number densities of two different species of particles contained in some bounded domain Ω in $\mathbb{R}^n$. Consider the functional $\mathcal{F}$ defined by

$$\begin{aligned} \mathcal{F}(m,n) = &\int_\Omega m(x)\ln m(x)\mathrm{d}x + \int_\Omega n(x)\ln n(x)\mathrm{d}x \\ &+ \beta \int_\Omega \int_\Omega J(|x-y|)m(x)n(y)\mathrm{d}x\mathrm{d}y\ . \end{aligned} \tag{5.1}$$

Here, J is a non negative, decreasing and compactly supported function on $\mathbb{R}_+$ with range R. Notice that we must impose more conditions on J in the case of two species that we did in the single component model. The reasons for this will be made clear in Section 6.

The problem considered in [6] is to minimize $\mathcal{F}(m,n)$ subject to the constraint that

$$\frac{1}{|\Omega|}\int_\Omega m(x)\mathrm{d}x \qquad \text{and} \qquad \frac{1}{|\Omega|}\int_\Omega n(x)\mathrm{d}x$$

have certain prescribed values. As shown in [6], this system undergoes a phase transition: The entropy terms in $\mathcal{F}$ would prefer to have m and n to be uniform. However, the mutual repulsion between the two species encourages them to segregate. For large values of β, the advantages of segregation can dominate, and the fluid separates into two phases, one rich in particles of species 1, and the other rich in particles of species 2. Our concern here is with the profiles of the densities at transition between the two phases.

The nature of the two phases is determined by considering the zero range model, in which the length scale R of the interaction J is negligible compared to the size of Ω. One is then led to consider the function

$$f_{\beta,\mu_1,\mu_2}(m,n) = m\ln m + n\ln n + \beta\hat{J}mn - \mu_1 m - \mu_2 n$$

which is a local free energy density. Here, $\widehat{J} = \int_{\mathbb{R}^n} J(|x|)\mathrm{d}x$, and μ_1 and μ_2 are Lagrange multipliers that enforce the constraint on the total particle numbers.

In [6] it is proved that, if $\mu_1 \neq \mu_2$ there is an unique couple $\bar{m}, \bar{n}$ minimizing $f_{\beta,\mu_1,\mu_2}(m,n)$. However, if $\mu_1 = \mu_2$, there is a β_c such that, if $\beta \leq \beta_c$ the minimizer is still unique, while, if $\beta > \beta_c$ there are densities $\rho^- < \rho^+$ such that the couples (ρ^+,ρ^-) and (ρ^-,ρ^+) are both minimizers of $f_{\beta,\mu_1,\mu_2}(m,n)$. We focus on the last case. Thus, in what follows $\mu_1 = \mu_2 = \mu$.

Analysis of the zero range model suffices to determine the quantity of the fluid that is present in each phase, but not the surface tension across the boundary. We now turn to the variational problem that determines the density profiles across the interface, and the surface tension.

First, we need to introduce the one dimensional version of J. Choose coordinate (x,y) on $\mathbb{R}^n$ with $x \in R$ and $y \in \mathbb{R}^{n-1}$, and define $\bar{J}$ on $\mathbb{R}$ by

$$\bar{J}(x) = \int_{\mathbb{R}^{n-1}} J(\sqrt{x^2 + |y|^2})\mathrm{d}y \ .$$

Notice that

$$\int_{\mathbb{R}} \bar{J}(x)\mathrm{d}x = \widehat{J} \ .$$

Let

$$g_{\beta,\mu} = \inf_{m,n\geq 0} f_{\beta,\mu,\mu}(m,n)$$

By what has been noted above,

$$g_{\beta,\mu} = f_{\beta,\mu,\mu}(\rho^-,\rho^+) = f_{\beta,\mu,\mu}(\rho^+,\rho^-) \ .$$

The functional $\mathcal{G}$ defined by

$$\mathcal{G}(m,n) = \int_{\mathbb{R}} \left[m(x)\ln m(x) + n(x)\ln n(x) + \beta \int_{\mathbb{R}} \bar{J}(x-y)m(x)n(y)\mathrm{d}y - g_{\beta,\mu} \right] \mathrm{d}x$$

is the *excess free energy* at a front. We look for the minimizers of this functional for $\beta > \beta_c$. The minimum value gives the surface tension across the phase boundary.

Our goal in the next sections is to prove a strict displacement convexity property of this excess free energy functional, and to show, as a consequence, the uniqueness of the minimizing fronts up to translation. As in the one component case, a rearrangement inequality will enable us to restrict our attention to monotone profiles. Let $\mathcal{M}_{\rho^-,\rho^+}$ be the subset of $\mathcal{C}_{\rho^-,\rho^+}$ consisting of monotone increasing profiles, let $\mathcal{M}_{\rho^+,\rho^-}$ be the subset of $\mathcal{C}_{\rho^+,\rho^-}$ consisting of monotone decreasing profiles

Our main goal mathematically in what follows is to show that the functional

$$(m,n) \mapsto \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \bar{J}(x-y)m(x)n(y)\mathrm{d}y - \hat{J}\rho^+\rho^- \right] \mathrm{d}x$$

is displacement convex on $\mathcal{M}_{\rho^-,\rho^+} \times \mathcal{M}_{\rho^+,\rho^-}$.

Then, taking the limit in which $\bar{J}(x)\mathrm{d}x$ tends to $\widehat{J}$ times a Dirac mass at the origin, and using our results from the single component case, we deduce that the functional

$$(m,n) \mapsto \int_{\mathbb{R}} \left[m(x)\ln m(x) + n(x)\ln n(x) - g_{\beta,\mu} + \hat{J}\rho^+\rho^- \right] \mathrm{d}x$$

is displacement convex on $\mathcal{M}_{\rho^-,\rho^+} \times \mathcal{M}_{\rho^+,\rho^-}$.

We shall prove the displacement convexity results in the next section. This time, we shall require certain moment conditions to obtain the displacement convexity. Hence, before we can apply these results, we need to show *a priori* that all minimizers have good localization properties. We do this by an analysis of the Euler Lagrange equation.

5.1. Convexity of the interaction energy for $\mathcal{G}$

Consider the functional $\mathcal{I}$ on $\mathcal{M}_{a,b} \times \mathcal{M}_{c,d}$ by

$$\mathcal{I}(m_1, m_2) = \int_{\mathbb{R}} \mathrm{d}x \left[\int_R J(x-y)m_1(x)m_2(y)\mathrm{d}y - \hat{J}\hat{m}(x)\hat{n}(x) \right] . \qquad (5.2)$$

We assume J to be non negative, even and compactly supported on $\mathbb{R}$ with range R. As before, we define $\widehat{J}$ to be the total mass of J, and we define

$$\widehat{m}(x) = \begin{cases} b & \text{for } x \geq 0 \ , \\ a & \text{for } x < 0 \end{cases} \qquad \text{and} \qquad \widehat{n}(x) = \begin{cases} d & \text{for } x \geq 0 \ , \\ c & \text{for } x < 0 \ . \end{cases}$$

Note that in the special case $a = d = \rho^-$ and $b = c = \rho^+$,

$$\mathcal{I}(m,n) = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \bar{J}(x-y)m(x)n(y)\mathrm{d}y - \hat{J}\rho^+\rho^- \right] \mathrm{d}x \ ,$$

but there is no extra effort required to treat the general case, and since it may prove useful, we do so here.

The first step in our analysis is to rewrite $\mathcal{I}$ as a functional on probability densities. Let the probability densities ρ_1 and ρ_2 be defined by

$$m_1(x) = a + (b-a)\int_{-\infty}^x \rho_1(t)dt; \qquad m_2(x) = c + (d-c)\int_{-\infty}^x \rho_2(t)dt \ . \quad (5.3)$$

To do this, we integrate by parts. Formally, one moves an antiderivative from each of ρ_1 and ρ_2 over to J. Since J is positive, integrating it twice produces a convex function W. This is indeed what happens, but one must be careful about the boundary terms. The boundary terms do not vanish, but as we shall see, they depend on the densities in a very nice way, and altogether, one obtains the desired displacement convexity.

To carry out this analysis, define

$$W(x) = \begin{cases} \int_0^x \left(\int_0^t J(s)\mathrm{d}s \right) \mathrm{d}t & \text{for } x > 0\ , \\ W(-x) & \text{for } x < 0\ . \end{cases} \tag{5.4}$$

Then $W''(x) = J(x)$, $W(0) = 0$, and W is an even convex function. Furthermore,

$$\lim_{x\to\infty} W'(x) = \frac{\hat{J}}{2}, \quad W(x) = \alpha + \frac{\hat{J}}{2}|x| \text{ for } |x| \geq R\ . \tag{5.5}$$

Theorem 5.1. *Let $m_1 \in \mathcal{M}_{a,b}$ and $m_2 \in \mathcal{M}_{c,d}$. Let ρ_1 and ρ_2 be the corresponding probability densities defined in (5.3). Then, provided ρ_1 and ρ_2 have finite first moments,*

$$\begin{aligned} \mathcal{I}(m_1, m_2) = {} & (a-b)(d-c) \int_{\mathbb{R}} \int_{\mathbb{R}} W(x-y)\rho_1(x)\rho_2(y)\mathrm{d}x\mathrm{d}y \\ & + [2(b-a)(d-c) + bc + ad]\alpha \\ & - \frac{\hat{J}}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} x\ \big[(b+a)(d-c)\rho_2(x) + (b-a)(c+d)\rho_1(x)\big]\mathrm{d}x\ . \end{aligned} \tag{5.6}$$

Note that $(a-b)(d-c) > 0$. Thus, $(a-b)(d-c)W(z)$ is a convex function of z on all of $\mathbb{R}$. It follows in the usual way that the first term on the right is displacement convex. Since W is strictly convex on the support of J, it follows as in the proof of Theorem 3.2 that this part of the functional is in fact strictly convex apart from translation.. The second term on the right is a constant. The third term is a linear combination of the first moments of ρ_1 and ρ_2. Since these first moments are displacement affine, we see that altogether, $\mathcal{I}(m_1, m_2)$ is strictly displacement convex, apart from translation.

The fact that Theorem 5.1 requires a conditions on first moments, while Theorem 3.2 does not, means that it will be a little more work to apply Theorem 5.1: We shall need an *a priori* estimate guaranteeing that for any critical point (m, n) of $\mathcal{G}$, the corresponding densities have finite first moments. We shall return to this after first proving the theorem.

Proof: We start considering the integral in x first, on a bounded interval $[-L, L]$. Since $J(x-y) = -\frac{\partial^2}{\partial x \partial y} W(x-y)$ we have that

$$
\begin{aligned}
&-\int_{-L}^{L} \frac{\partial^2}{\partial x \partial y} W(x-y) m_1(x) \mathrm{d}x = \int_{-L}^{L} \frac{\partial}{\partial y} W(x-y)(b-a)\rho_1(x) \mathrm{d}x \\
&-\frac{\partial}{\partial y} W(L-y) m_1(L) + \frac{\partial}{\partial y} W(-L-y) m_1(-L)
\end{aligned}
\tag{5.7}
$$

Moreover,

$$
\begin{aligned}
&\int_{-L}^{L}\int_{-L}^{L} J(x-y) m_1(x) m_2(y) \mathrm{d}y = \int_{-L}^{L}\int_{-L}^{L} \frac{\partial}{\partial y} W(x-y)(b-a)\rho_1(x) m_2(y) \mathrm{d}x \mathrm{d}y \ + \\
&\int_{-L}^{L} \left[-\frac{\partial}{\partial y} W(L-y) m_1(L) + \frac{\partial}{\partial y} W(-L-y) m_1(-L) \right] m_2(y) \mathrm{d}y
\end{aligned}
\tag{5.8}
$$

Now we integrate by parts once more, this time in y:

$$
\begin{aligned}
&\int_{-L}^{L} \frac{\partial}{\partial y} W(x-y) m_2(y) \mathrm{d}y = -\int_{-L}^{L} W(x-y)(d-c)\rho_2(y) \mathrm{d}y \\
&+ W(x-L) m_2(L) - W(x+L) m_2(-L) \ .
\end{aligned}
\tag{5.9}
$$

Summarizing,

$$
\begin{aligned}
&\int_{-L}^{L}\int_{-L}^{L} J(x-y) m_1(x) m_2(y) \mathrm{d}y \\
&= -(b-a)(d-c) \int_{-L}^{L}\int_{-L}^{L} W(x-y)\rho_1(x)\rho_2(y) \mathrm{d}x \mathrm{d}y \\
&+ \int_{-L}^{L} \Big[-\frac{\partial}{\partial y} W(L-y) m_1(L) + \frac{\partial}{\partial y} W(-L-y) m_1(-L) \Big] m_2(y) \mathrm{d}y \\
&+ (b-a) \int_{-L}^{L} \left[W(x-L) m_2(L) - W(x+L) m_2(-L) \right] \rho_1(x) \mathrm{d}x
\end{aligned}
\tag{5.10}
$$

Let us examine the boundary terms

$$
B_1 := \int_{-L}^{L} \left[-\frac{\partial}{\partial y} W(L-y) m_1(L) + \frac{\partial}{\partial y} W(-L-y) m_1(-L) \right] m_2(y) \mathrm{d}y \ ,
$$

$$
B_2 := (b-a) \int_{-L}^{L} \left[W(x-L) m_2(L) - W(x+L) m_2(-L) \right] \rho_1(x) \mathrm{d}x
$$

We have

$$B_1 = m_1(L)\int_{-L}^{L} W(L-y)(d-c)\rho_2(y)\mathrm{d}y - m_1(-L)\int_{-L}^{L} W(-L-y)(d-c)\rho_2(y)\mathrm{d}y$$

$$+m_1(L)\left[-W(L-y)m_2(y)\right]_{-L}^{+L} + m_1(-L)\left[W(-L-y)m_2(y)\right]_{-L}^{+L} .$$

$$= (d-c)\int_{-L}^{L} \left[m_1(L)W(L-y) - m_1(-L)W(-L-y)\right]\rho_2(y)\mathrm{d}y$$

$$+m_1(L)\left[-W(0)m_2(L) + W(2L)m_2(-L)\right] + m_1(-L)\left[W(2L)m_2(L) - W(0)m_2(-L)\right]$$

For $2L > R$, where R is the range of the interaction J, the last two terms give

$$(bc+ad)(\hat{J}L+\alpha) + \mathcal{O}(1)$$

To compute the other term, we consider, for a function f rapidly decaying, $\int_{-L}^{L} f(x)W(x+L)\mathrm{d}x$ and $\int_{-L}^{L} f(x)W(x-L)\mathrm{d}x$. We have

$$\int_{-L}^{L} f(x)W(x+L)\mathrm{d}x = \int_{0}^{2L} f(z-L)W(z)\mathrm{d}z = \int_{0}^{R} f(z-L)W(z)\mathrm{d}z + \int_{R}^{2L} f(z-L)(\frac{\hat{J}}{2}z+\alpha)\mathrm{d}z$$

The first term vanishes in the limit $L \to \infty$ because of the decay of f and of the boundedness of $W(z)$ for $z \in [0, R]$. The second term becomes, if $\int_{\mathbb{R}} |x| f(x)\mathrm{d}x < \infty$,

$$\int_{R-L}^{L} f(x)(\frac{\hat{J}}{2}(x+L)+\alpha)\mathrm{d}x = \frac{\hat{J}}{2}\int_{\mathbb{R}} xf(x)\mathrm{d}x + (\alpha + \frac{\hat{J}}{2}L)\int_{\mathbb{R}} f(x)\mathrm{d}x + \mathcal{O}(1)$$

In conclusion,

$$\int_{-L}^{L} f(x)W(x\pm L)\mathrm{d}x = \pm\frac{\hat{J}}{2}\int_{\mathbb{R}} xf(x)\mathrm{d}x + (\alpha + L\frac{\hat{J}}{2})\int_{\mathbb{R}} f(x)\mathrm{d}x + \mathcal{O}(1)$$

Now we apply this result to B_2, where the decaying function is ρ_1, to get

$$B_2 = (b-a)\left[-(c+d)\frac{\hat{J}}{2}\int_{\mathbb{R}} x\rho_1(x)\mathrm{d}x + \alpha(d-c)\int_{\mathbb{R}}\rho_1(x)\mathrm{d}x + \frac{\hat{J}}{2}L(d-c)\int_{\mathbb{R}}\rho_1(x)\mathrm{d}x\right] + \mathcal{O}(1)$$

Now we apply to B_1:

$$B_1 = (d-c)\left[-(b+a)\frac{\hat{J}}{2}\int_{\mathbb{R}} x\rho_2(x)\mathrm{d}x + \right. \tag{5.11}$$

$$\left. \alpha(b-a)\int_{\mathbb{R}}\rho_2(x)\mathrm{d}x + \frac{\hat{J}}{2}L(b-a)\int_{\mathbb{R}}\rho_2(x)\mathrm{d}x\right] + (bc+ad)(\hat{J}L+\alpha) + \mathcal{O}(1)$$

Finally,

$$B_1 + B_2 - \hat{J}\int_{\mathbb{R}} \hat{m}_1(x)\hat{m}_2(x)\mathrm{d}x = [2(b-a)(d-c) + bc + ad]\alpha$$

$$-\frac{\hat{J}}{2}(b+a)(d-c)\int_{\mathbb{R}} y\rho_2(y)\mathrm{d}y - \frac{\hat{J}}{2}(b-a)(c+d)\int_{\mathbb{R}} x\rho_1(x)\mathrm{d}x + \mathcal{O}(1)$$

□

We close this section with a corollary showing that one could also use Theorem 5.1 to prove displacement convexity of the interaction energy in the one component model, albeit under slightly more restrictive hypothesis.

Corollary 5.2 *Let J and W defined as in the Theorem 5.1. Let m be a function that increases monotonically from $-m_\beta$ to m_β. Let ρ denote m', the derivative of m. Consider the functional $\Phi(m)$ given by*

$$\Phi(m) = \int_R \int_R J(x-y)\Big[m(x)m(y) - m_\beta^2\Big]\mathrm{d}x\mathrm{d}y \ .$$

Then

$$\Phi(m) = -\int_R \int_R W(x-y)\rho(x)\rho(y)\mathrm{d}x\mathrm{d}y - 6\alpha m_\beta^2 \ .$$

Proof. The functional $\Phi(m)$ is equal to $-\mathcal{G}(m_1, m_2)$ by putting $m_1(x) = m(x)$ and $m_2(x) = -m(x)$.

This shows that $-\Phi$ is strictly displacement convex, up to translation.

6. Properties of the minimizers of $\mathcal{G}$.

We restrict our attention to the case $a = d$, $b = c$, as in our binary fluid model. We need two results on the minimizers for $\mathcal{G}$, the first of which allows us to restrict our attention to monotone profiles when seeking to minimize $\mathcal{G}$. The second guarantees the existence of moments for the two densities corresponding to any minimizing pair (m, n). These theorems are:

Theorem 6.1. *Suppose that $J(x)$ is even non negative and decreasing. Then any minimizer (m_1, m_2) of $\mathcal{G}(m_1, m_2)$ in $\mathcal{C}_{\rho^-,\rho^+} \times \mathcal{C}_{\rho^+,\rho^-}$ is monotone in the sense that m_1 is increasing and m_2 is decreasing.*

This theorem makes it easy to establish the existence of minimizers for G. The minimizers satisfy an Euler–Lagrange equation form which we can deduce *a priori* estimated needed to apply Theorem 5.1.

Theorem 6.2. *Suppose that $J(x)$ is even non negative and decreasing on $\mathbb{R}_+$. Any minimizer $w = (w_1, w_2)$ of $\mathcal{G}$ in $\mathcal{C}_{\rho^-,\rho^+} \times \mathcal{C}_{\rho^+,\rho^-}$ satisfies*

$$\rho^- < w_i(x) < \rho^+$$

for any $x \in \mathbb{R}$. It has derivative almost everywhere which is strictly positive and with $\|w_i'\|_{L^1(\mathbb{R})}$ is bounded. Furthermore, it satisfies the Euler-Lagrange equations

$$\ln m(x) + \beta(J * n)(x) = \mu, \quad \ln n(x) + \beta(J * m)(x) = \mu, \tag{6.1}$$

where $\mu = \mu_1 - 1$ and $*$ denotes convolution. Its derivative w satisfies almost everywhere the equations

$$\frac{w_1'(x)}{w_1(x)} + \beta(J * w_2')(x) = 0, \quad \frac{w_2'(x)}{w_2(x)} + \beta(J * w_1')(x) = 0 \tag{6.2}$$

Finally, it converges to its asymptotic values exponentially fast, in the sense that there is $\alpha > 0$ such that

$$(w_1(x) - \rho_{\mp})e^{\alpha|x|} \to 0 \text{ as } x \to \mp\infty, \quad (w_2(x) - \rho_{\pm})e^{\alpha|x|} \to 0 \text{ as } x \to \mp\infty.$$

The proof of Theorem 6.1 is adapted from a related result in [6] for functions on the d dimensional torus. One could instead adapt the proof of Alberti's rearrangement inequality in [1] and remove the requirement that J be decreasing. But the present approach has the advantage of working also on the torus, and not only the line. The proof of the final part of Theorem 6.2, which is important for our application here since it provides the existence of moments, is adapted from the proof of a similar result for the one component system in [10]. In the rest of this section, we present these proofs.

Proof of Theorem 6.1: To show this, we use a rearrangement inequality similar to those introduced in [6] for the analogous problem in the d-dimensional torus. For any $x_0 \in \mathbb{R}$, let T_{x_0} denote the reflection about x_0:

$$T_{x_0}(x) = 2x_0 - x \ .$$

Then define $\mathcal{D}$, as the set of functions on $\mathbb{R}$ having finite limits at $\pm\infty$ and the operators $R^{\pm}_{x_0}$ on $\mathcal{D}$ by

$$R^+_{x_0} g(x) = \begin{cases} \max\{g(x), g(T_{x_0})\} & \text{if } x \geq x_0 \ , \\ \min\{g(x), g(T_{x_0})\} & \text{if } x \leq x_0. \end{cases} \tag{6.3}$$

$$R^-_{x_0} h(x) = \begin{cases} \max\{h(x), h(T_{x_0})\} & \text{if } x \leq x_0 \ , \\ \min\{h(x), h(T_{x_0})\} & \text{if } x \geq x_0. \end{cases} \tag{6.4}$$

Let us also define

$$\hat{g}(x) = \begin{cases} \lim_{x \to -\infty} g(x) & \text{if } x < 0 \\ \lim_{x \to +\infty} g(x) & \text{if } x \geq 0 \end{cases}$$

and $\hat{h}$ similarly.

For any fixed x_0 and $g, h \in \mathcal{D}$, let $g^\star$ denote $R^+_{x_0} g$ and $h_\star = R^-_{x_0} h$. We now wish to show that

$$\int_{\mathbb{R}} \left[\int_{\mathbb{R}} g(x)J(x-y)h(y)\mathrm{d}y - \hat{J}\hat{g}(x)\hat{h}(x) \right] \mathrm{d}x \geq \int_{\mathbb{R}} \left[\int_{\mathbb{R}} g^\star(x)J(x-y)h_\star(y)\mathrm{d}y - \hat{J}\hat{g}(x)\hat{h}(x) \right] \mathrm{d}x$$

with equality if and only if $g = g^\star$ and $h = h^\star$.

To do this, let $\mathbb{H}_+$ denote the half line $\{x \mid x > x_0\}$, and $\mathbb{H}_-$ denote the half line $\{x \mid x < x_0\}$ and observe that

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{R}} g(x)J(x-y)h(y)\mathrm{d}x\mathrm{d}y = \\ &\int_{\mathbb{H}_+} \int_{\mathbb{H}_+} g(x)J(x-y)h(y)\mathrm{d}x\mathrm{d}y + \int_{\mathbb{H}_-} \int_{\mathbb{H}_-} g(x)J(x-y)h(y)\mathrm{d}x\mathrm{d}y + \\ &\int_{\mathbb{H}_-} \int_{\mathbb{H}_+} g(x)J(x-y)h(y)\mathrm{d}x\mathrm{d}y + \int_{\mathbb{H}_+} \int_{\mathbb{H}_-} g(x)J(x-y)h(y)\mathrm{d}x\mathrm{d}y = \\ &\int_{\mathbb{H}_+} \int_{\mathbb{H}_+} g(x)J(x-y)h(y)\mathrm{d}x\mathrm{d}y + \int_{\mathbb{H}_+} \int_{\mathbb{H}_+} g(T_{x_0}x)J(x-y)h(T_{x_0}y)\mathrm{d}x\mathrm{d}y + \\ &\int_{\mathbb{H}_+} \int_{\mathbb{H}_+} g(T_{x_0}x)J(T_{x_0}x-y)h(y)\mathrm{d}x\mathrm{d}y + \int_{\mathbb{H}_+} \int_{\mathbb{H}_+} g(x)J(x-T_{x_0}y)h(T_{x_0}y)\mathrm{d}x\mathrm{d}y \end{aligned} \tag{6.5}$$

The desired inequality is then a consequence of the following inequality for pairs of real numbers: Let a_1 and a_2 and b_1 and b_2 be any four positive real numbers. Rearrange a_1 and a_2 to decrease, and b_1 and b_2 to increase; i.e., let $a_1^\star = \max\{a_1, a_2\}$, $a_2^\star = \min\{a_1, a_2\}$, $b_1^\star = \min\{b_1, b_2\}$ and $b_2^\star = \max\{b_1, b_2\}$. Then

$$a_1^\star b_1^\star + a_2^\star b_2^\star - a_1 b_1 - a_2 b_2 = \Delta \ \leq 0, \tag{6.6}$$

$$a_1^\star b_2^\star + a_2^\star b_1^\star - a_1 b_2 - a_2 b_1 = -\Delta \ \geq 0, \tag{6.7}$$

and there is equality if and only if $a_1 = a_1^\star$ and $b_1 = b_1^\star$ or $a_1^\star = a_2$ and $b_1^\star = b_2$.

We now apply the above inequalities with

$$a_1 = g(x) \quad a_2 = g(T_{x_0}x) \quad b_1 = h(y) \quad \text{and} \quad b_2 = h(T_{x_0}y)\ . \tag{6.8}$$

Then

$$a_1^\star = R^+_{x_0} g(x) \quad a_2^\star = R^+_{x_0} g(T_{x_0}x) \quad b_1^\star = R^-_{x_0} h(y) \quad \text{and} \quad b_2^\star = R^-_{x_0} h(T_{x_0}y)\ . \tag{6.9}$$

Since

$$J(T_{x_0}x - y) = J(x - T_{x_0}y) < J(x-y),$$

we get

$$\begin{gathered} g(x)J(x-y)h(y)+g(T_{x_0}x)J(x-y)h(T_{x_0}y)+ \\ g(T_{x_0}x)J(T_{x_0}x-y)h(y)+g(x)J(T_{x_0}x-y)h(T_{x_0}y)- \\ R^+_{x_0}g(x)J(x-y)R^-_{x_0}h(T_{x_0}y)+R^+_{x_0}g(T_{x_0}x)J(x-y)R^-_{x_0}h(T_{x_0}y)- \\ R^+_{x_0}g(T_{x_0}x)J(T_{x_0}x-y)R^-_{x_0}h(y)+R^+_{x_0}g(x)J(T_{x_0}x-y)R^-_{x_0}h(T_{x_0}y)= \\ -\Delta\big[J(x-y)-J(x-T_{x_0}y)\big]\geq 0 \end{gathered}$$

for almost every x and y in $\mathbb{H}_+$, with equality if and only if

$$g(T_{x_0}x)\leq g(x) \qquad \text{and} \qquad h(T_{x_0}y)\geq h(y) \tag{6.10}$$

or

$$g(T_{x_0}x)\geq g(x) \qquad \text{and} \qquad h(T_{x_0}y)\leq h(y) \tag{6.11}$$

for almost every x and y in $\mathbb{H}_+$. Now unless g is constant, we can find x and x_0 so that either $g(T_{x_0}x) < g(x)$ or $g(T_{x_0}x) > g(x)$. Suppose it is the first case. Then (6.10) holds, and for almost every y, we must have $h(T_{x_0}y) \geq g(y)$. Making a similar argument for h, we see that one of (6.10) or (6.11) must hold for almost every x and y. The only way that this can happen is if g and h are monotone. Now, by integrating (6.10) on $\mathbb{H}_+$ we conclude the proof. □

Proof of Theorem 6.2: Everything but the exponential decay is standard, and details of the proofs of similar results can be found in [6]. To prove the exponential decay, we once again take advantage of the finite range R of J.

Define a transformation $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Phi(m,n) = (e^{\mu-\beta\widehat{J}n}, e^{\mu-\beta\widehat{J}m}) \ .$$

Then (ρ^+,ρ^-) and (ρ^-,ρ^+) are two stable fixed points of Φ; the Jacobian of Φ, $D\Phi$, is a strict contraction at either of them. Thus, there is a $\delta > 0$ and an $\epsilon > 0$ so that if

$$|m-\rho^+| + |n-\rho^-| < \delta \quad \Rightarrow \quad \|D\phi(m,n)\| < 1-\epsilon \ .$$

Now, consider any minimizer $w = (w_1, w_2)$ with $\lim_{x\to\infty} w_1(x) = \rho^+$ and $\lim_{x\to\infty} w_2(-x) = \rho^-$. Then there is an $L < \infty$ so that

$$x \geq L \quad \Rightarrow \quad |w_1(x)-\rho^+| + |w_2(-x)-\rho^-| < \delta \ .$$

Now for $x > L + R$,

$$\frac{J}{\widehat{J}} * w_1(x) \geq \rho^+ - \delta \qquad \text{and} \qquad \frac{J}{\widehat{J}} * w_2(x) \leq \rho^- + \delta \ .$$

Since

$$(w_1(x), w_2(x)) = \Phi\left(\frac{J}{\widehat{J}} * w_1(x), \frac{J}{\widehat{J}} * w_2(x)\right) \ ,$$

it follows that for $x > L + R$,

$$|w_1(x) - \rho^+| + |w_2(-x) - \rho^-| < (1 - \epsilon)\delta \ .$$

Iterating this argument leads to the conclusion that for $x > L + kR$,

$$|w_1(x) - \rho^+| + |w_2(-x) - \rho^-| < (1 - \epsilon)^k\delta \ .$$

A similar argument applies as x tends to $-\infty$. □

7. For the functional $\mathcal{G}$, critical points are minimizers

We are now ready to prove the main theorem for $\mathcal{G}$:

Theorem 7.1. *If (w_1, w_2) and (v_1, v_2) are any two critical points of $\mathcal{G}$ in $\mathcal{M}_{\rho^-,\rho^+} \times \mathcal{M}_{\rho^+,\rho^-}$, then there is an $a \in R$ so that*

$$(v_1(x), v_2(x)) = (w_1(x - a), w_2(x - a)) \ . \tag{7.1}$$

Thus, there is exactly one critical point (w_1, w_2) such that $w_1(0) = w_2(0)$. It is symmetric in the sense that $w_1(x) = w_2(-x)$ for all x.

Proof: Theorem 5.1 is applicable since by Theorem 6.2, the corresponding densities have finite moments of every order. Thus, by Theorem 5.1, the interaction part of the $\mathcal{G}$, as is strictly displacement convex. Then remaining term is simply a linear combination of functions of m and n to which we can apply Theorem 2.1.

Now, if (m_λ, n_λ) is the displacement convex interpolation between (w_1, w_2) and (v_1, v_2), $\mathcal{G}(m_\lambda, n_\lambda)$ is constant since both endpoints are critical points. By the strict convexity up to translation, we see that (7.1) is true.

Since $(w_1, w_2) \in \mathcal{M}_{\rho^-,\rho^+} \times \mathcal{M}_{\rho^+,\rho^-}$, and both functions are strictly monotonic, there is some b such that $w_1(b) = w_2(b)$. Because of the strict monotonicity of w_1; i.e., the strict positivity of its derivative, which was proved in Theorem 6.2, this value of b is unique.

Next, by the symmetries of the functional, since $(w_1(x), w_2(x))$ is any minimizer of $\mathcal{G}$ in $\mathcal{M}_{\rho^-,\rho^+} \times \mathcal{M}_{\rho^+,\rho^-}$, then so is $(w_2(-x), w_1(-x))$. Hence, by the first part of the Theorem, there is an $a \in R$ so that

$$(w_2(-x), w_1(-x)) = (w_1(x - a), w_2(x - a)) \ . \tag{7.2}$$

Evaluating both sides at $x = 0$, we see that since $w_1(0) = w_2(0)$, $w_1(-a) = w_2(-a)$. By the uniqueness of the crossing point established above, $a = 0$, so that

$$(w_2(-x), w_1(-x)) = (w_1(x), w_2(x))$$

for all x. □

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